What is Induction?

Induction is a nifty (and sometimes confusing) proof technique that enables you to prove that a given property is true about all natural numbers. For example, we could show that for all \( n \geq 1, \sum_{i=1}^{n} i = n(n+1)/2, \) or that \( (x^n - 1) \) is divisible by \( (x - 1). \) What is so interesting about this proof technique is that it can prove properties about an infinite set (the natural numbers), and yet the technique itself is so simple.

The basic idea is this: you prove that a statement holds for all natural numbers \( n \) by showing (1) that it holds for a base case, such as \( n = 0, \) then showing (2) that if the statement holds for \( n = k \) for some \( k \) (this assumption is called the \textit{induction hypothesis}), then it holds for \( n = k + 1. \)

Why does this work? Since you’ve shown the base case, you know the property is true for \( n = 0. \) Moreover, you’ve shown that if the property holds for \( n = 0 \) then it also holds for \( n = 1; \) therefore, the base case implies it is true for \( n = 1. \) You can use the step (2) again to show that it is true for \( n = 2, \) then \( n = 3, \) and so on, as long as you want.

Some students have trouble with the idea of induction because the inductive step appears to assume what you are trying to prove, yet this isn’t the case. You’re not assuming it’s true; you are proving that if it were true for some smaller problem then it would be true for the larger. In combination with your base case, this will prove that it is true for all sizes of the problem.

Dominoes

Suppose you set up some dominoes in a long, long hallway, such that each domino is close enough to the previous one so that they knock each other down. I ‘accidentally’ knock over the first domino. I now claim all the dominoes will fall.

Why? Well I’ve specifically set up the dominoes so that if the \( k^{th} \) domino falls then it knocks down the \( k + 1^{st} \) domino, for any \( k \geq 1. \) Notice that this doesn’t actually guarantee that the dominoes fall until I actually knock down the first domino; at this point, we start the chain reaction that knocks down all dominoes.

Let \( P(n) \) be the statement that the \( n^{th} \) domino falls. Then I want to show that \( P(n) \) is true for all integers \( n \geq 1. \) We prove this by induction on \( n, \) the number of dominoes.

First off, we know that the first domino gets knocked down because that’s the one action we take. Thus, we’ve shown that when \( n = 1, P(n) \) is true.

Now, suppose that the \( k^{th} \) domino falls, that is, suppose that \( P(k) \) is true for some \( k \geq 1. \) Does this imply that the \( k + 1 \)st domino falls? Yes; this is specifically how we set up the dominoes. Therefore, if the \( k^{th} \) domino falls, so will the \( k + 1^{st} \). Thus \( P(k) \) indeed implies \( P(k + 1) \) (alternatively stated as: if \( P(k) \) is true for some \( k \geq 1 \) then \( P(k + 1) \) is also true).

And we’re done! Because we know \( P(1) \) and that \( P(k) \) implies \( P(k + 1) \) for all \( k \geq 1, \) we know that \( P(n) \) is true for all \( n \geq 1. \)
Sum of an Arithmetic Series

Let $P(n)$ be the statement that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We want to show that $P(n)$ is true for all $n \geq 1$.

**Base Case:** We want to show that $P(n)$ is true for $n = 1$: that is, that $\sum_{i=1}^{1} i = \frac{1(2)}{2}$. Fortunately,

left-hand side (LHS) = $\sum_{i=1}^{1} i = 1$

= $(1 + 2)/2 = $ right-hand side (RHS).

**Induction Hypothesis:** Assume $P(n)$ is true for some $n = k \geq 1$. That is, assume that

$\sum_{i=1}^{k} i = k(k + 1)/2$.

**Note:** The induction hypothesis is assuming precisely want you want to prove, but for some specific $k$, that is, you only have to replace all $n$’s with $k$’s in your $P(n)$ statement.

**Inductive Step:** With the assumption that $P(n)$ is true for some $n = k$, we want to prove that $P(n)$ is true for $n = k + 1$, that is, we want to prove that if $P(k)$ is true then $P(k + 1)$ is also true. Specifically, to show $P(k + 1)$ is true, we need to prove $\sum_{i=1}^{k+1} i = (k + 1)(k + 2)/2$.

$LHS = \sum_{i=1}^{k+1} i$

= $(k + 1) + \sum_{i=1}^{k} i$

= $(k + 1) + k(k + 1)/2$ by the induction hypothesis

= $2(k + 1)/2 + k(k + 1)/2$

= $(k + 1)(k + 2)/2$

= RHS.

**Note:** We must use our induction hypothesis somewhere in this step. If we do not use it anywhere, then we are able to prove $P(k + 1)$ is true without the assumption that $P(k)$ is true, which means that $P(k + 1)$ is always true, for any $k \geq 1$...

**Conclusion:** By mathematical induction, we have shown that for all $n \geq 1$, $\sum_{i=1}^{n} i = n(n + 1)/2$. 
Main Steps

Please follow these 5 steps when doing a proof by induction.

**Step 1: State your** $P(n)$. State your what property of $n$ you are trying to prove, which should be a property as a function of $n$. Also state for which $n$ you will prove your $P(n)$ to be true (is it for all $n \geq 0$? $n \geq 1$? $n \geq 100$?).

**Step 2: State your base case.** State for which $n$ your base case is true, and prove it. Typically, this will be the smallest $n$ for which you are trying to prove $P(n)$, but occasionally you’ll need more than one base case.

**Step 3: State your induction hypothesis.** State your induction hypothesis! Without it, the whole proof falls apart. Usually it is just restating your $P(n)$ but with $n = k$ (everywhere you see an $n$, put a $k$ instead).

**Step 4: Inductive Step.** Now consider $P(n)$ where $n = k + 1$. This is where you try to prove a larger case of the problem than you assumed in your induction hypothesis. What are you trying to prove? Keep this in mind when you do this step. Remember, use your induction hypothesis somewhere, and clearly state where. If you haven’t used your induction hypothesis in the step, then you are not doing a proof by induction. So you’d better need to use it.

**Step 5: Conclusion.** This is optional. You can re-state the property you were trying to prove.

Comments

- If your $P(n)$ doesn’t mention $n$ in it anywhere, then $P$ isn’t a property of $n$ and that’s worrisome.

- $P(n)$ is a *property*, not a number, so you cannot manipulate it mathematically, like $P(n) = 5$, or $P(n + 1) < P(n)$.

- Be careful with the base case... sometimes you will need more than one, as with some recurrence relations.

- When trying to prove some equation holds, that is, if you are trying to prove that $x = y$ for some $x$ and $y$, please do not start with assuming they are equal and then modifying both sides of the equations until you get an equation that is actually true. For example:

\[
\begin{align*}
x &= y \\
0 \times x &= 0 \times y \\
0 &= 0.
\end{align*}
\]

Obviously I can ‘prove’ that $x = y$ using this method for any $x$ and $y$, regardless of whether or not $x$ and $y$ are actually equal. What is better is to start with $x$, make modifications to $x$ through a string of equalities that somehow ends with $y$. That is, $x = \cdots = y$. 
Exercises

a) Prove that $1+2+4+\cdots+2^n = 2^{n+1}-1$, that is, $P(n)$ is the statement that $\sum_{i=0}^{n} 2^i = 2^{n+1}-1$.

b) Prove that $1+3+5+\cdots+(2n-1) = n^2$, that is, $P(n)$ is the statement that $\sum_{i=1}^{n} (2i-1) = n^2$.

c) Prove that $3n \geq 1 + 2n$ for all integers $n \geq 0$.

d) Show that $x^n - 1$ is divisible by $(x - 1)$ for all integers $n \geq 1$.

e) Show that $3^{4n} - 1$ is divisible by 80 for all integers $n \geq 1$.

f) Show that $\sin(x + 180n) = (-1)^n \sin x$ for all integers $n \geq 1$.

g) Show that $\sum_{i=1}^{n} \log \frac{i+1}{i} = \log(n+1)$ for $n \geq 1$.

h) Show that $n! > 2^n$ for all integers $n \geq 5$.

i) Show that $(1 + 2 + 3 + \cdots + n)^2 = \frac{1}{4} n^2 (n+1)^2$ for all integers $n \geq 1$.

j) Show that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$ for all integers $n \geq 1$. 