Motivation

Mathematical induction is a simple yet powerful technique that allows us to prove infinitely many statements with a finite amount of work. The concept of an inductive proof doesn’t actually require much mathematical background; one could realistically consider teaching induction in elementary school. Induction proofs are highly structured, yet leave plenty of room room for creativity.

So, you may say, this is great and all, but this isn’t a course in mathematics so why do we care? This course is about problem solving, and when you solve a problem, it is quite important that you can convince yourself and others that it does what it is supposed to do. Induction is one tool to help you argue about the correctness of your algorithms (and therefore your programs). This is the first time in cs 150 that we will discuss such correctness arguments; there is a whole course (cs 280) whose main focus is problem solving techniques and accompanying correctness arguments.

In this document, we use induction to determine how to best cut a pizza with a Samurai sword, how to solve the towers of Hanoi problem, how to tile a chess board with tri-ominoes, how to make the most of your jelly bean eating experience, and many other useful skillz!

A Ladder Metaphor

Suppose there is a very long ladder on the side of a very tall tower. The rungs on this ladder are such that from any one rung you are able to reach the next rung. Given this information, can you reach the top of the ladder?

Your answer ought to be “it depends”: it depends on whether the very first rung is reachable or not! If you are able to reach the first rung, then because the rungs are close together, you are able to reach the second, and then the third, and then the fourth, and so on. Conversely, if you are not able to reach the first rung then you are stuck on the ground.

Humour me and think about this ladder situation in another way. Let’s label the rungs of the ladder with numbers, such that the bottom rung is labelled 0 and the labels increase by 1 as we go up the ladder. Let $P(n)$ be the property that we are able to reach the $n$th rung. You can (and should) think of $P$ as a boolean function, that takes a single integer parameter $n$, and returns true if you can reach the $n$th rung, and false otherwise.

With this notation, the statement “the rungs are such that from any one rung you are able to reach the next rung” is the same as “for any $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true.” This doesn’t say anything about the reachability of the first rung; that is, we don’t know if $P(0)$ is true. If $P(0)$ is true, though, then we can apply the “if $P(k)$ is true then $P(k + 1)$ is also true” for $k = 0$ and get that $P(1)$ is true, which then implies that $P(2)$ is true, and so on, so that $P(n)$ is true for all $n \geq 0$. Conversely, if $P(0)$ is not true, then we cannot say anything about the validity of $P(n)$ for any $n \geq 1$. 
Induction Introduction

The principles of induction follow that of the ladder. If we are trying to prove that we can reach the \( n \)th rung of a ladder, for all \( n \), we first have to prove that our rungs are close enough so that if we can reach the \( k \)th rung then we can reach the \( k + 1 \)st. This isn’t sufficient to prove the reachability of all rungs, though; we must also prove that we can reach the first rung.

Of course, we won’t be proving statements about ladders! Instead, we prove properties about natural numbers, using the same ladder idea. You’ll have some boolean property \( P(n) \) that you are trying to prove for some infinite set of natural numbers (for example, for all \( n \geq 0 \) or for all \( n \geq 1 \)). To prove \( P(n) \) for all requisite \( n \), you’ll prove two things: (1) that if \( P(k) \) holds for some fixed \( k \), then \( P(k+1) \) also holds, and (2) that \( P(0) \) also holds (or \( P(1) \), or some other small number). This combination will prove that \( P(n) \) holds for the infinitely many \( n \) that you specified.

Why does this work? You’ve explicitly shown the property is true for \( n = 0 \). Moreover, you’ve shown that if the property holds for \( k = 0 \) (which it does, because you just did that) then it also holds for \( k = 1 \); therefore, the base case implies it is true for \( n = 1 \). You can use the step (1) again to show that it is true for \( n = 2 \), then \( n = 3 \), and so on, infinitely, as long as you want.

Here is another argument. Suppose that you’ve shown (1) and (2), yet there is some number such that the property is not true. Let \( k + 1 \) be the smallest number for which the property isn’t true. How did the property become false? Well, if it’s false for \( k + 1 \), then it must also be false for \( k \) (because of (1)). But \( k + 1 \) was the smallest number for which the property was false, which is a contradiction. Therefore (1) and (2) imply there is no such \( k \).

Enough abstract jibber jabber. It’s time for an example.

Example 1: Sum of Squares

Suppose we are trying to come up with a formula for the sum of the first \( n \) squares, for \( n \geq 0 \). We may try a few numbers out to see if we can find a pattern:

\[
\begin{align*}
0^2 &= 0 \\
0^2 + 1^2 &= 1 \\
0^2 + 1^2 + 2^2 &= 5 \\
0^2 + 1^2 + 2^2 + 3^2 &= 14
\end{align*}
\]

Okay, I’m not sure if you’re seeing a pattern, but it looks like the sum of the first \( n \) squares may be \( n(n+1)(2n+1)/6 \). It certainly works for the few examples above, but I’d like to prove that the formula works for every single \( n \geq 0 \), that is, I want to prove an infinite number of equalities. Let’s see how to do this with induction:
Step 0. For all $n \geq 0$, we want to show that the sum of the first $n$ squares is equal to $n(n + 1)(2n + 1)/6$.

Step 1. For any $n \geq 0$, let $P(n)$ be the property that

$$0^2 + 1^2 + 2^2 + \cdots + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$ 

We want to show that $P(n)$ is true for all $n \geq 0$.

Step 2. As a base case, consider when $n = 0$. We will show that $P(0)$ is true: that is, that $0^2 + \cdots + 0^2 = 0(0 + 1)(2 \cdot 0 + 1)/6$. Fortunately,

left-hand side $= 0^2 + \cdots + 0^2 = 0 = 0(0 + 1)(0 + 1)/6 = $ right-hand side.

Step 3. For the induction hypothesis, suppose (hypothetically) that $P(k)$ were true for some fixed $k \geq 0$. That is, suppose that

$$0^2 + 1^2 + 2^2 + \cdots + (k - 1)^2 + k^2 = \frac{k(k + 1)(2k + 1)}{6}.$$ 

Step 4. Now we prove that $P(k + 1)$ is true, using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that

$$0^2 + 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = \frac{(k + 1)(k + 2)(2(k + 1) + 1)}{6}.$$ 

Step 5. The proof that $P(k + 1)$ is true (given that $P(k)$ is true) is as follows:

left-hand side $= 0^2 + 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2$

$= (0^2 + 1^2 + 2^2 + \cdots + k^2) + (k + 1)^2$

$= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2$ by the induction hypothesis $P(k)$

$= \frac{k(k + 1)(2k + 1) + 6(k + 1)^2}{6}$ by algebra

$= \frac{(k + 1)(2k^2 + k) + (k + 1)(6k + 6)}{6}$ by algebra

$= \frac{(k + 1)(2k^2 + 7k + 6)}{6}$ by algebra

$= \frac{(k + 1)(k + 2)(2k + 3)}{6}$ by algebra

$= $ right-hand side.

Therefore we have shown that if $P(k)$ is true, then $P(k + 1)$ is also true, for any $k \geq 0$.

Step 6. The steps above have shown that for any $k \geq 0$, if $P(k)$ is true, then $P(k + 1)$ is also true. Combined with the base case, which shows that $P(0)$ is true, we have shown that for all $n \geq 0$, $P(n)$ is true, as desired.
7 Step Process

As you may have gathered, induction proofs follow a 7-step structure, as described below.

Step 0: State (the infinite set of) statements that you want to prove.

For all $n \geq \langle\text{base case}\rangle$, we want to show that ___________.

where the blank should contain some property of $n$.

Step 1: State your $P(n)$. State what property of $n$ you are trying to prove, which should be a property as a function of $n$. Also state for which $n$ you will prove your $P(n)$ to be true (is it for all $n \geq 0$? $n \geq 1$? $n \geq 100$?).

For any $n \geq \langle\text{base case}\rangle$, let $P(n)$ be the property that ___________.

where the blank should contain some property of $n$, possibly as it was written in Step 0.

Step 2: State your base case. State for which $n$ your base case is true, and prove it. Typically, this will be the smallest $n$ for which you are trying to prove $P(n)$ (i.e. what you used as $\langle\text{base case}\rangle$), but occasionally you'll need more than one base case.

As a base case, consider when $n = \langle\text{base case}\rangle$. We will show that $P(\langle\text{base case}\rangle)$ is true, by: ___________.

Step 3: State your induction hypothesis. State your induction hypothesis. Here you are usually just hypothetically asserting the property $P$ for some fixed $k \geq \langle\text{base case}\rangle$.

For the induction hypothesis, suppose (hypothetically) that $P(k)$ were true for some fixed $k \geq \langle\text{base case}\rangle$, that is, suppose that ___________.

where the blank should contain the statement $P(k)$.

Step 4: State your inductive assertion. State that you will prove $P(k + 1)$ given the (hypothetical) assumption that $P(k)$ is true.

Now we prove that $P(k+1)$ is true, using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that ___________.

Step 5: Prove the inductive step. Here is where you actually do the proof that $P(k + 1)$ is true, and thus this is where the creativity comes in. This proof is going to be different for each induction proof; sometimes it will use algebra, number theory, common sense, etc. It will always need to use the induction hypothesis $P(k)$, and you should clearly label when and where you use this assumption. If you haven’t used your induction hypothesis, then you aren’t doing a proof by induction (and you should be worried!)

The proof that $P(k + 1)$ is true (given that $P(k)$ is true) is as follows: ___________.

Step 6: Conclusion.

We have shown that if $P(k)$ is true, then $P(k + 1)$ is true. Thus, because $P(\langle\text{base case}\rangle)$ is true, you have that $P(n)$ is true for all $n \geq \langle\text{base case}\rangle$. 
Comments

- Check that you $P(n)$ mentions $n$ in it somewhere, and that it doesn’t mention other variables. Remember, $P$ is just like a method, that takes in one variable $n$. You should use that variable in $P$, and you should use others.

- $P(n)$ is a boolean property, not a number, so you cannot manipulate it mathematically, like $P(n) = 5$, or $P(n + 1) < P(n)$.

- Be careful with the base case... sometimes you will need more than one, as with some recurrence relations.

- You must use your induction hypothesis somewhere in the proof of the inductive step, otherwise you are not doing a proof by induction. Check to be sure.

- If you’re stuck in your inductive step, take a step back for a moment. What are you trying to prove? Keep this in mind when you work on proving $P(k + 1)$. Since you have to use your induction hypothesis somewhere, you may want to think about how you can manipulate what you’ve got into something that resembles your induction hypothesis. Also, you may want to check any algebra you’ve done as that can often be the source of problems!

- When trying to prove some equation holds, that is, if you are trying to prove that

\[
\text{left-hand-side} = \text{right-hand-side}
\]

for some left-hand-side and right-hand-side, please do not start with assuming they are equal and then modifying both sides of the equations until you get an equation that is actually true. For example:

\[
\begin{align*}
\text{left-hand-side} &= \text{right-hand-side} \\
0 \times \text{left-hand-side} &= 0 \times \text{right-hand-side} \\
0 &= 0.
\end{align*}
\]

Therefore they are equal.

Obviously I can ‘prove’ that left-hand-side = right-hand-side using this method for any left-hand-side and right-hand-side, regardless of whether or not they are actually equal! What is better is to start with left-hand-side, make modifications to left-hand-side through a string of equalities that somehow ends with right-hand-side. That is,

\[
\begin{align*}
\text{left-hand-side} &= \cdots \\
&= \cdots \\
&= \text{right-hand-side}.
\end{align*}
\]

This will guarantee that you don’t prove something that isn’t true.
Example 2: Maximizing Slices of Pizza

Suppose you have a pizza and a Samurai sword. With \( n \) (straight-line) cuts of the sword, what is the maximum number of slices you can create?

**Step 0.** For all \( n \geq 0 \), we want to show that \( n \) cuts can produce \( 1 + n(n + 1)/2 \) slices of pizza.

**Step 1.** For any \( n \geq 0 \), let \( P(n) \) be the property that \( n \) cuts can produce \( 1 + n(n + 1)/2 \) slices of pizza.

**Step 2.** As a base case, consider when \( n = 0 \). We will show that \( P(0) \) is true: with 0 cuts, we have exactly one pizza-shaped slice of pizza, and right-hand side = \( 1 + 0(0 + 1)/2 = 1 \).

**Step 3.** For the induction hypothesis, suppose (hypothetically) that \( P(k) \) is true for some fixed \( k \geq 0 \). That is, suppose that \( k \) cuts can produce \( 1 + k(k + 1)/2 \) slices of pizza.

**Step 4.** Now we prove that \( P(k + 1) \) is true, using the (hypothetical) induction assumption that \( P(k) \) is true. That is, we prove that \( k + 1 \) cuts can produce \( 1 + (k + 1)(k + 2)/2 \) slices.

**Step 5.** The proof that \( P(k + 1) \) is true (given that \( P(k) \) is true) is as follows:

First we observe that \( k + 1 \) sword cuts can be seen as \( k \) sword cuts, and then one final cut. Each time this final cut passes through a pre-existing slice, it splits that slice in half. We’re trying to maximize the number of new slices this final cut makes, and so we want it to intersect as many of the previous slices as possible.

As we make the last cut, new slices are created by cutting through an existing slice. As the final cut transitions from one existing slice to another, it crosses through an existing sword cut that can only be crossed once (because all cuts are straight). Therefore, the last cut can at most intersect each of the previous \( k \) cuts exactly once, passing through (and bisecting) at most \( k + 1 \) of the pre-existing slices (one before and after all first and last existing sword cut, respectively, and the \( k - 1 \) slices in between the \( k \) cuts.) Therefore

\[
\text{left-hand side} = \max \# \text{slices with } k + 1 \text{ cuts} \\
= \max \# \text{slices with } k \text{ cuts} + \max \# \text{slices made by last cut} \\
= 1 + k(k + 1)/2 + \max \# \text{slices made by last cut} \\
= 1 + k(k + 1)/2 + (k + 1) \\
= 1 + \frac{k(k + 1) + 2(k + 1)}{2} \\
= 1 + \frac{(k + 1)(k + 2)}{2} \\
= \text{right-hand side.}
\]

Therefore we have shown that if \( P(k) \) is true, then \( P(k + 1) \) is also true.

**Step 6.** The steps above show that for any \( k \geq 0 \), if \( P(k) \) is true, then \( P(k + 1) \) is also true. Combined with the base case, which shows that \( P(0) \) is true, we have shown that for all \( n \geq 0 \), \( P(n) \) is true, as desired.
Example 3: Towers of Hanoi

There are $n$ disks of different diameters stacked from largest to smallest on a start tower. The goal is to move all disks to an end tower, such that they are stacked in same order. You have one extra tower, and you are restricted to moving one disk at a time, such that you never place a larger disk on top of a smaller disk. We propose the following algorithm:

```markdown
tower( n disks, start tower, end tower, extra tower)
  if n == 0, do nothing and return.
  recursively move top n-1 disks from start to extra, using end as third tower
    (that is, call tower(n-1, start tower, extra tower, end tower) )
  move the nth disk from the start tower to the end tower
  recursively move the n-1 disks on extra to end, using start as third tower
    (that is, call tower(n-1, extra tower, end tower, start tower) )
```

**Step 0.** For all $n \geq 0$, we want to show that the algorithm moves $n$ disks from the start tower to the end tower, following the rules, in $2^n - 1$ steps, such that larger disks are never on smaller disks.

**Step 1.** For any $n \geq 0$, let $P(n)$ be the property that the algorithm moves $n$ disks from the start tower to the end tower, following the rules, in $2^n - 1$ steps.

**Step 2.** As a base case, consider when $n = 0$. We will show that $P(0)$ is true, that is, that the algorithm moves 0 disks in $2^1 - 1 = 0$ moves, following the rules. And this is what it does.

**Step 3.** For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq 0$. That is, suppose that the algorithm moves $k$ disks from the start tower to the end tower, following the rules, in $2^k - 1$ steps.

**Step 4.** Now we prove that $P(k+1)$ is true, using our (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that the algorithm moves $k + 1$ disks from the start tower to the end tower, following the rules, in $2^{k+1} - 1$ steps.

**Step 5.** The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows: We have to prove that the algorithm uses $2^{k+1} - 1$ steps, and that each of these steps is “valid”. For the steps:

- left-hand side = \# steps to move $k + 1$ disks
  - $= 2(\# \text{ steps to move } k \text{ disks}) + 1$ by the algorithm definition
  - $= 2(2^k - 1) + 1$ by the IH
  - $= 2^{k+1} - 1 = \text{right-hand side}$

As for the validity of the steps, the algorithm on $k + 1$ towers leaves the largest tower on the bottom of the start tower, then makes a recursive call on the smallest $k$ towers. All of these moves are (by the IH) valid, and never move the largest disk. We then make one valid move, of this largest disk from the start tower to an empty tower, then again recursively move the smallest $k$ disks (valid, by the IH).

Therefore we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

**Step 6.** The steps above have shown that for any $k \geq 0$, if $P(k)$ is true, then $P(k + 1)$ is also true. Combined with the base case, which shows that $P(0)$ is true, we have shown that for all $n \geq 0$, $P(n)$ is true, as desired.
Example 4: Jelly Beans

You have \( n \geq 0 \) jelly beans, each of a different flavour. As you know, different subsets of jelly bean flavours produce different taste bud sensations. How many different flavour combinations are there? (We’ll count no flavour, as in, no jelly beans, as a distinct flavour combination.)

**Step 0.** For all \( n \geq 0 \), we want to show that there are \( 2^n \) flavour combinations of \( n \) jelly beans.

**Step 1.** For any \( n \geq 0 \), let \( P(n) \) be the property that there are \( 2^n \) flavour combinations of \( n \) jelly beans.

**Step 2.** As a base case, consider when \( n = 0 \). We will show that \( P(0) \) is true: with 0 jelly beans, there is only one flavour combination (the boring, no flavour one), and \( 1 = 2^0 \).

**Step 3.** For the induction hypothesis, suppose (hypothetically) that \( P(k) \) is true for some fixed \( k \geq 0 \). That is, suppose that there are \( 2^k \) flavour combinations of \( k \) jelly beans.

**Step 4.** Now we prove that \( P(k+1) \) is true, using our (hypothetical) induction assumption that \( P(k) \) is true. That is, we prove that there are \( 2^{k+1} \) flavour combinations of \( k + 1 \) jelly beans.

**Step 5.** Now we prove \( P(k+1) \) is true, using the fact that \( P(k) \) is true:

Consider your \( k + 1 \) beans, and suppose kiwi is one of your flavours. Then the total number of flavour combinations is equal to the flavour combinations that include kiwi, and those that do not. The number of flavour combinations that do not contain kiwi is precisely the number of flavour combinations of \( k \) jelly beans. And for each combination that *does* include kiwi corresponds to a unique subset of the remaining \( k \) jelly beans, and thus the number of combinations that contain kiwi is also exactly the number of flavour combinations of \( k \) jelly beans. More precisely, we have:

\[
\text{left-hand side} = \# \text{ combinations of } k + 1 \text{ beans} \\
= \# \text{ combinations of } k + 1 \text{ beans that contain kiwi} + \\
\# \text{ combinations of } k + 1 \text{ beans that don’t contain kiwi} \\
= \# \text{ combinations of } k \text{ beans (other than kiwi)} + \\
\# \text{ combinations of } k \text{ beans (that don’t contain kiwi)} \\
= 2^k + 2^k \\
= 2^{k+1} \quad \text{by the IH} \\
= \text{right-hand side.}
\]

Therefore we have shown that if \( P(k) \) is true, then \( P(k+1) \) is also true.

**Step 6.** The steps above have shown that for any \( k \geq 0 \), if \( P(k) \) is true, then \( P(k+1) \) is also true. Combined with the base case, which shows that \( P(0) \) is true, we have shown that for all \( n \geq 0 \), \( P(n) \) is true, as desired.
Example 5: Sum of Triplet Products

**Step 0.** For all \( n \geq 1 \), we want to show that the sum of the first \( n \) “triplet products”, i.e. 
\[
(1)(2)(3) + (2)(3)(4) + (3)(4)(5) + \cdots + n(n+1)(n+2),
\]
is equal to \( \frac{n(n+1)(n+2)(n+3)}{4} \).

**Step 1.** For any \( n \geq \), let \( P(n) \) be the property that

**Step 2.** As a base case, consider when \( n = \). We will show that \( P(\quad) \) is true:

**Step 3.** For the induction hypothesis, suppose (hypothetically) that \( P(k) \) were true for some fixed \( k \geq \). That is, suppose that

**Step 4.** Now we will prove that \( P(k+1) \) is true, using the (hypothetical) induction assumption that \( P(k) \) is true. That is, we must prove that

**Step 5.** The proof that \( P(k+1) \) is true (given that \( P(k) \) is true) is as follows: Therefore we have

shown that if \( P(k) \) is true, then \( P(k+1) \) is also true.

**Step 6.** The steps above have shown that for any \( k \geq \), if \( P(k) \) is true, then \( P(k+1) \) is also true. Combined with the base case, which shows that \( P(1) \) is true, we have shown that for all \( n \geq \), \( P(n) \) is true, as desired.
Example 6: Tiling with Triominoes

For any $n \geq 0$, consider a $2^n \times 2^n$ checkerboard, with one corner removed. Can you always tile the checkerboard with L-shaped “tri-ominoes”? If so, how?

**Step 0.** For all $n \geq \quad$, we want to show that

**Step 1.** For any $n \geq \quad$, let $P(n)$ be the property that

**Step 2.** As a base case, consider when $n = \quad$. We will show that $P(\quad)$ is true:

**Step 3.** For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq \quad$. That is, suppose that

**Step 4.** Now we will prove that $P(k + 1)$ is true, using our (hypothetical) induction assumption that $P(k)$ is true. That is, we must prove that

**Step 5.** The proof that $P(k + 1)$ is true (given that $P(k)$ is true) is as follows:

Therefore we have shown that if $P(k)$ is true, then $P(k + 1)$ is also true.

**Step 6.** The steps above have shown that for any $k \geq \quad$, if $P(k)$ is true, then $P(k + 1)$ is also true. Combined with the base case, which shows that $P(\quad)$ is true, we have shown that for all $n \geq \quad$, $P(n)$ is true, as desired.
Example 7: Grasshopper

On the first cell of a strip sits a grasshopper. Each minute it jumps to the right either to the next cell or to the second to next cell. Find the number of ways it can reach the $n$th cell.

**Step 0.** For all $n \geq 0$, we want to show that the grasshopper can reach the $n$th cell in $\text{fib}(n)$ ways.

**Step 1.** For any $n \geq 0$, let $P(n)$ be the property that

Step 2. As a base cases, consider when $n = 0, 1$. We will show that $P(0)$ and $P(1)$ is true:

**Step 3.** For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq 0$. That is, suppose that

**Step 4.** Now we will prove that $P(k + 1)$ is true, using our (hypothetical) induction assumption that $P(k)$ is true. That is, we must prove that

**Step 5.** Now we will prove $P(k + 1)$ is true, using the fact that $P(k)$ is true:

Therefore we have shown that if $P(k)$ is true, then $P(k + 1)$ is also true.

**Step 6.** The steps above have shown that for any $k \geq 0$, if $P(k)$ is true, then $P(k + 1)$ is also true. Combined with the base case, which shows that $P(0)$ is true, we have shown that for all $n \geq 0$, $P(n)$ is true, as desired.
Induction Template

Step 0. For all $n \geq \underline{\quad}$, we want to show that 

\[ \underline{\quad} \]

Step 1. For any $n \geq \underline{\quad}$, let $P(n)$ be the property that 

\[ \underline{\quad} \]

Step 2. As a base case, consider when $n = \underline{\quad}$. We will show that $P(\underline{\quad})$ is true:

\[ \underline{\quad} \]

Step 3. For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq \underline{\quad}$. That is, suppose that 

\[ \underline{\quad} \]

Step 4. Now we will prove that $P(k+1)$ is true, using our (hypothetical) induction assumption that $P(k)$ is true. That is, we must prove that 

\[ \underline{\quad} \]

Step 5. The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows:

\[ \underline{\quad} \]

Therefore we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

Step 6. The steps above have shown that for any $k \geq \underline{\quad}$, if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(\underline{\quad})$ is true, we have shown that for all $n \geq \underline{\quad}$, $P(n)$ is true, as desired.