the worst case. See the notes on this chapter (Section 19.5.2) for a brief discussion of research concerning other cost-sharing approaches.

19.4 Facility Location

In the models we have considered so far, players construct networks so as to achieve certain connectivity-based goals. Intuitively, these goals are meant to capture players' desires to provide service for some implicit population of network users. Given this perspective, we might then ask what happens when we instead view players as financially motivated agents; after all, service providers are primarily concerned with maximizing profits, and only maintain networks for this purpose. This suggests a model in which players not only build networks but also charge for usage, while network users spur competition by seeking the cheapest service available.

We will consider here a pricing game introduced by Vetta (2002) that is based on the facility location problem. In the facility location problem, we want to locate \( k \) facilities, such as Web servers or warehouses, so as to serve a set of clients profitably. Our focus here will be to understand the effect of selfish pricing on the overall efficiency of the networks that players form.

We first present Vetta's competitive facility location problem, in which players place facilities so as to maximize their own profit. We then show that this facility location game is a potential game, and prove that the price of anarchy for an even broader class of games is small.

19.4.1 The Model

Suppose that we have a set of users that need a service, and \( k \) service providers. We assume that each service provider \( i \) has a set of possible locations \( A_i \) where he can locate his facility.

Define \( A = \bigcup_i A_i \) to be the set of all possible facility locations. For each location \( s_i \in A_i \), there is an associated cost \( c_{js} \) for serving customer \( j \) from location \( s_i \). We can think of these costs as associated with edges of a bipartite graph that has all users on one side and all of \( A \) on the other, as shown on Figure 19.4. A strategy vector

![Figure 19.4. The bipartite graph of possible locations and clients. Selected facilities are marked in black.](image-url)
s = \{s_1, \ldots, s_k\} can be thought of as inducing a subgraph of this graph consisting of the customers and the selected location nodes (marked as black on Figure 19.4).

Our goal is to maximize social welfare, rather than simply minimizing the cost of the constructed network. We assume that customer \( j \) has a value \( \pi_j \) for service, and gathers \( \pi_j - p \) benefit by receiving service at a price \( p < \pi_j \). Locating a facility \( s_i \) is free, but that service provider \( i \) must pay \( c_{js_i} \) to serve client \( j \) from location \( s_i \). Doing so generates a profit of \( p - c_{js_i} \). If provider \( i \) services customer \( j \) from location \( s_i \), then this arrangement creates a social value (or surplus) of \( \pi_j - c_{js_i} \), the value \( \pi_j \) of service minus the cost \( c_{js_i} \) at which the service is provided. Note that this social surplus is independent of the price \( p = p_{ij} \) charged; varying \( p_{ij} \) simply redistributes welfare between the customer and the provider. We define the social welfare \( V(S) \) to be the total social value over all providers and customers.

To simplify notation, we assume that \( \pi_j \geq c_{js_i} \) for all \( j, i \), and \( s_i \in A_i \). To see that this requires no loss of generality, note that decreasing \( c_{js_i} \) to be at most \( \pi_j \) does not change the value of any assignment: when \( \pi_j < c_{js_i} \) customer \( j \) cannot be served from location \( s_i \), while \( \pi_j = c_{js_i} \) allows us to serve customer \( j \) from location \( s_i \) at cost. In either case, the assignment of serving client \( j \) from facility \( s_i \) results in 0 social value.

To complete the game, we must specify how prices are set and assignments are determined. Given a strategy vector \( s \), we assume that each customer is assigned to a facility that can serve for the lowest cost. The price \( p_{ij} \) charged to a customer \( j \) using player \( i \)'s facility \( s_i \) is the cost of the second cheapest connection available to \( j \), i.e., \( \min_{j' \neq j} c_{js_i} \). Intuitively, this is the highest price \( i \) could expect to get away with charging \( j \); charging any more would give some player \( i' \) an incentive to undercut \( i \).

Indeed, we can construct an equivalent interpretation of this game in which prices are selected strategically. Consider a three-stage game where both providers and customers are strategic agents. In the first stage, providers select facility locations. In the second stage, providers set prices for users. And, in the last stage, users select a provider for service, and pay the specified price.

As we saw in Chapter 1, subgame perfect equilibrium is a natural solution concept for multistage games. We will use here a further refinement of this concept, the trembling hand perfect equilibrium for extensive form games (see Mas-Colell et al., 1995). Assume that with probability \( \epsilon > 0 \), each player picks a strategy chosen uniformly at random, and chooses a best strategy with the remaining \( (1 - \epsilon) \) probability. We use the notion of subgame perfect equilibrium for this \( \epsilon \)-perturbed game. A trembling hand perfect equilibrium is an equilibrium that can be reached as the limit of equilibria in the \( \epsilon \)-perturbed game as \( \epsilon \) approaches 0. This stronger notion of stability is required to prevent providers from offering unprofitably low prices and thereby forcing other providers to artificially lower their own prices.

### 19.4.2 Facility Location as a Potential Game

We start by proving that the facility location game is a potential game.

**Theorem 19.16** The facility location game is a potential game with social value \( V(s) \) as the potential function.
**Proof** We need to argue that if a provider \( i \) changes her selected location, then the change in social welfare \( V(s) \) is exactly the change in the provider's welfare. To show this, we imagine provider \( i \) choosing to "drop out of the game" and show that the change in social welfare \( V(s) \) is exactly \( i \)'s profit.

If provider \( i \) "drops out," each client \( j \) that was served by provider \( i \) switches over to his second best choice. Recall that \( p_{ij} \) is exactly the cost of this choice. Thus the client will be served at cost \( p_{ij} \) rather than \( c_{ijn} \), so the increase in cost is \( p_{ij} - c_{ijn} \), exactly the profit provider \( i \) gathers from \( j \).

To prove the statement about provider \( i \) changing his strategy, we can think of the change in two steps: first the provider leaves the game, and then reenters with a different strategy. The change in social welfare is the difference between the profit of provider \( i \) in the two strategies. \( \square \)

**Corollary 19.17** There exists a pure strategy equilibrium, and furthermore, all efficient outcomes of the facility location game are stable. Thus, the price of stability is 1. Finally, best response dynamics converge to an equilibrium, but this equilibrium may not be socially optimal.

Our next goal is to prove that the price of anarchy for this facility location game is small. However, it turns out that the proof applies to a much broader class of games, which we present now.

### 19.4.3 Utility Games

Vetta (2002) introduced the facility location game as one example of a large class of games called utility games. In a utility game, each player \( i \) has a set of available strategies \( A_i \), which we will think of as locations, and we define \( A = \bigcup_i A_i \). A social welfare function \( V(S) \) is defined for all \( S \subseteq A \). Observe that welfare is purely a function of the selected locations, as is the case with the facility location game. In defining the socially optimum set, we will consider only sets that contain one location from each strategy set \( A_i \). However, various structural properties of the function \( V(S) \) will be assumed for all \( S \subseteq A \). For a strategy vector \( s \), we continue to use \( V(s) \) as before, and let \( \alpha_i(s) \) denote the welfare of player \( i \). A game defined in this manner is said to be a utility game if it satisfies the following three properties.

1. \( V(S) \) is submodular: for any sets \( S \subseteq S' \subseteq A \) and any element \( s \in A \), we have \( V(S' - s) - V(S) \geq V(S' + s) - V(S') \). In the context of the facility location game, this states that the marginal benefit to social welfare of adding a new facility diminishes as more facilities are added.
2. The total value for the players is less than or equal to the total social value: \( \sum \alpha_i(s) \leq V(s) \).
3. The value for a player is at least his added value for the society: \( \alpha_i(s) \geq V(s) - V(s_{-i}) \).

A utility game is basic if property (iii) is satisfied with equality, and monotone if all \( S \subseteq S' \subseteq A \), \( V(S) \leq V(S') \).
To view the facility location game as a utility game, we consider only the providers as players. We note that the social welfare \( V(S) = \sum_j (\pi_j - \min_{e \in S} e_{je}) \) is indeed purely a function of the selected locations.

**Theorem 19.18** The facility location problem is a monotone basic utility game.

**Proof** Property (ii) is satisfied essentially by definition, and we used the equality of property (iii) property in proving Theorem 19.16. To show property (i), notice that adding a new facility decreases the cost of serving some of the clients. The magnitude of this decrease can only become smaller if the clients are already choosing from a richer set of facilities. Finally, adding a facility cannot cause the cost of serving a client to increase, and thus the facility location game is monotone. \( \square \)

### 19.4.4 The Price of Anarchy for Utility Games

Since the facility location game is a potential game with the social welfare as the potential function, the price of stability is 1. In fact, this applies for any basic utility game (any utility game with \( \alpha_i(s) = V(s) - V(s - s_i) \) for all strategy vectors \( s \) and players \( i \)). Unfortunately, the increased generality of utility games comes at a cost; these games are not necessarily potential games, and indeed, pure equilibria do not always exist. However, we now show that for monotone utility games that do possess pure equilibria (such as the facility location game), the price of anarchy is at most 2.

**Theorem 19.19** For all monotone utility games the social welfare of any pure Nash equilibrium is at least half the maximum possible social welfare.

**Proof** Let \( S \) be the set of facilities selected at an equilibrium, and \( O \) be the set of facilities in a socially optimal outcome. We first note that \( V(O) \leq V(S \cup O) \) by monotonicity. Let \( O^i \) denote the strategies selected by the first \( i \) players in the socially optimal solution. That is, \( O^0 = \emptyset, O^1 = \{o_1\}, \ldots , O^k = O \). Now

\[
V(O) - V(S) \leq V(S \cup O) - V(S) = \sum_{i=0}^{n} [V(S \cup O^i) - V(S \cup O^{i-1})].
\]

By submodularity (property (i))

\[
V(S \cup O^i) - V(S \cup O^{i-1}) \leq V(S + o_i - s_i) - V(S - s_i)
\]

for all \( i \). Using property (iii), we can further bound this by \( \alpha_i(S + o_i - s_i) \). Since \( S \) is an equilibrium, \( \alpha_i(S + o_i - s_i) \leq \alpha_i(S) \). Together these yield

\[
V(O) - V(S) \leq V(O \cup S) - V(S) \leq \sum_i \alpha_i(S).
\]

Finally, property (ii) implies that \( \sum_i \alpha_i(S) \leq V(S) \), so \( V(O) \leq 2V(S) \), and hence the price of anarchy is at most 2. \( \square \)
19.4.5 Bounding Solution Quality without Reaching an Equilibrium

For any monotone basic utility game, one can also bound the quality of the solution without assuming that players reach an equilibrium, as was shown in a sequence of two papers by Mirrokni and Vetta (2004) and Goemans et al. (2005).

**Theorem 19.20** Consider an arbitrary solution in a monotone basic utility game. Suppose that at each time step, we select a player at random and make a best response move for that player. For any constant \( \epsilon > 0 \) the expected social value of the solution after \( O(n) \) such moves is at least \( 1/2 - \epsilon \) times the maximum possible social value.\(^1\)

**Proof** Let \( S \) be a state, and \( O \) be an socially optimal strategy vector. We will prove that the expected increase in social welfare in one step is at least \( \frac{1}{n}(V(O) - 2V(S)) \), which implies the claimed bound after \( O(n) \) steps.

Let \( \beta_i \) be the maximum possible increase in the value for player \( i \). Thus the expected increase in value is \( \frac{1}{n} \sum \beta_i \). Selecting strategy \( o_i \) is an available move, so \( \beta_i \geq \alpha_i(S - s_i + o_i) - \alpha_i(S) \), and by basicness, \( \beta_i \geq V(S - s_i + o_i) - V(S - s_i) \).

The rest of the proof mirrors the price of anarchy proof above. We have

\[
V(O) - V(S) \leq \sum_{i=0}^{n}[V(S - s_i + o_i) - V(S - s_i)]
\]

as before. We bound \( V(S + o_i - s_i) - V(S - s_i) \leq \alpha_i(S) + \beta_i \). Using this with property (ii) yields

\[
V(O) - V(S) \leq \sum_i (\alpha_i(S) + \beta_i) \leq V(S) + \sum_i \beta_i.
\]

Thus \( \sum \beta_i \geq V(O) - 2V(S) \), and the expected increase in \( V(S) \) is \( \frac{1}{n}(V(O) - 2V(S)) \). The difference \( V(O) - 2V(S) \) is expected to decrease by \( 1 - \frac{1}{2} \) each step. After \( n/2 \) steps, the difference is expected to decrease by \( 1 \) factor of \( \epsilon \), and after \( \log(\epsilon^{-1})n \) steps shrinks to an \( \epsilon \) factor. \( \square \)

19.5 Notes

19.5.1 Local Connection Game

Network formation games have a long history in the social sciences, starting with the work of Myerson (1977, 1991). A standard example of such games can be found in Jackson and Wolinsky (1996) (see Jackson (2006) for a more comprehensive survey). These network formation games are often used to model the creation of social networks, and aim to capture pairwise relations between individuals who may locally form direct links to one another. In other contexts, these games might model peering relations.

\(^1\) The constant in the \( O(.) \) notation depends on \( \log \epsilon^{-1} \).