

## Main Steps

The 4 main steps of a proof by contradiction are as follows:

**Step 1: State your goal.** State what property  $P$  you are trying to prove.

**Step 2: Assume the contrary.** Assume that  $P$  is not true, by way of contradiction.

**Step 3: Find the contradiction.** This is where the magic happens. Use the fact that  $P$  is not true to logically deduce a conclusion you know is incorrect. In this way you know the starting assumption could not be correct, and  $P$  must be true.

**Step 4: Conclusion.** This is optional. You can re-state the property  $P$ .

## Comments

- This technique often works when everything else fails.

## Example

Suppose we want to prove that  $\sqrt{2}$  is irrational.

By way of contradiction, suppose that  $\sqrt{2}$  is rational, that is, that

$$\sqrt{2} = \frac{p}{q},$$

for integers  $p, q$  without a common factor. Then

$$2q^2 = p^2,$$

and  $p^2$  is divisible by 2, thus both  $p^2$  and  $p$  are even numbers; we write  $p = 2k$  for some integer  $k$ . Now

$$2q^2 = 4k^2 .$$

Using the same argument we have that  $q^2$ , and hence  $q$ , is divisible by 2. But then both  $p$  and  $q$  have 2 as a common factor, contradicting our original assumption that  $p$  and  $q$  share no common factors.

Therefore no such  $p$  and  $q$  exist, and  $\sqrt{2}$  is irrational.

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