

## Main Steps

The 4 main steps for proving a language  $A$  is regular is as follows:

**Step 1: Construct a FA.** Construct a finite automaton  $M$  (either DFA or NFA) that you believe recognizes  $A$ , that is, you want  $L(M) = A$ .

**Step 2: Define!** Be sure to completely specify all necessary aspects of your finite automaton as formally as possible. Check that you have provided all 5 elements of the 5-tuple.

**Step 3: Prove your FA is correct.** For every  $x \in \Sigma^*$ , show that  $x \in L(M) \iff x \in A$ . This shows that your finite automaton accepts the desired language.

**Step 4: Prove Intermediate Steps.** Did you use a generalized version of  $\hat{\delta}$  that we haven't actually defined? Did you use some concatenation property that we haven't proven yet? To be complete, all such intermediate assumptions should have formal proofs.

## Comments

- You can possibly avoid this whole process by using closure properties of regular languages. In particular, do you know anything about the complement of  $A$ ? Is  $A$  the intersection of two languages you already know are regular? Think about this before you delve into the steps above.
  - Sometimes you will be building your FA out of another, different FA. See the example on the back of this sheet for an example!
  - Be explicit and precise in your definitions. The more complete you are in constructing the FA, the easier your proof will be.
  - It may help to informally describe your finite automaton in advance, using pebbles or other techniques. This can help focus what you're trying to do, and give me an idea as well.
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## Example

Given a language  $L$  over the alphabet  $\Sigma$ , we can define a new language

$$\text{Bloat}(L) = \{\sigma_1\sigma_2\sigma_3\sigma_4 \dots \sigma_{2n-1}\sigma_{2n} \mid \sigma_1\sigma_3 \dots \sigma_{2n-1} \in L \text{ and } \sigma_2, \sigma_4, \dots, \sigma_{2n} \in \Sigma\}.$$

Prove that if  $L$  is a regular language, then  $\text{Bloat}(L)$  is also regular.

*Proof.* If  $L$  is regular, then there is a DFA  $M = (Q, \Sigma, \delta, s, F)$  that accepts  $L$ , that is,  $L(M) = L$ . We will show  $\text{Bloat}(L)$  is regular by constructing a DFA  $M'$  that accepts  $\text{Bloat}(L)$ . Here is an informal description of  $M'$ . We split each state of  $q \in Q$  into two states  $q$  and  $q'$ . All transitions into  $q$  remain entering  $q$ . All transitions leaving  $q$  now leave  $q'$ . We can transition from  $q$  to  $q'$  on any input symbol. In this way, we can insert any letter  $a \in \Sigma$  where previously there was none. We start in state  $s'$  so that we do not start with an inserted symbol, and we want to end in any  $f'$  such that  $f \in F$  in order to end with an inserted symbol.

Now let's do this more formally. Define the DFA  $M' = (Q', \Sigma, \delta', s', F')$  as follows.

$$\begin{aligned} Q' &= Q \cup \{q' \mid q \in Q\} \\ \delta'(q, a) &= q' \quad \forall a \in \Sigma \\ \delta'(q', a) &= p \text{ if } \delta(q, a) = p \\ F' &= \{f' \mid f \in F\} \end{aligned}$$

To complete the proof, we need to show this DFA actually recognizes the language  $\text{Bloat}(L)$ . We do this by showing that a string  $x$  is accepted by  $M'$  iff  $x \in \text{Bloat}(L)$ . To this end, consider any  $x \in L(M')$ . We will show that  $x \in \text{Bloat}(L)$  using a sequence of iff's.

$$\begin{aligned} x \in L(M') &\iff \hat{\delta}'(s', x) \in F' \\ &\iff \exists \text{ states } q'_0, q_1, q'_1, q_2, q'_2, \dots, q'_{k-1}, q_k, q'_k \in Q' \text{ such that } q'_0 = s', q'_k \in F', \\ &\quad \delta'(q'_i, \sigma_{2i+1}) = q_{i+1}, \delta'(q_i, \sigma_{2i}) = q'_i, \text{ and } x = \sigma_1\sigma_2 \dots \sigma_{2k} \\ &\iff \exists \text{ states } q_0, q_1, q_2, \dots, q_k \in Q \text{ such that } \delta(q_i, \sigma_{2i+1}) = q_{i+1}, q_k \in F, \\ &\quad \sigma_2, \sigma_4, \dots, \sigma_{2k} \in \Sigma \text{ and } x = \sigma_1\sigma_2 \dots \sigma_{2k} \\ &\iff \hat{\delta}(q_0, \sigma_1\sigma_3\sigma_5 \dots \sigma_{2k-1}) \in F, \sigma_2, \sigma_4, \dots, \sigma_{2k} \in \Sigma, \text{ and } x = \sigma_1\sigma_2 \dots \sigma_{2k} \\ &\iff x = \sigma_1\sigma_2 \dots \sigma_{2k} \text{ such that } \sigma_1\sigma_3\sigma_5 \dots \sigma_{2k-1} \in L \text{ and } \sigma_2, \sigma_4, \dots, \sigma_{2k} \in \Sigma \\ &\iff x \in \text{Bloat}(L) \end{aligned}$$