CSCI 383 Fall 2011 **Proof by Induction**

Main Steps

Please follow these 4 steps when doing a proof by induction.

Step 1: State your $P(n)$. State what property of n you are trying to prove, which should be a property as a function of n. Also state for which n you will prove your $P(n)$ to be true.

Step 2: State your base case. State for which n your base case is true, and prove it. Typically, this will be the smallest n for which you are trying to prove $P(n)$, but occasionally you'll need more than one base case.

Step 3: State your induction hypothesis. State your induction hypothesis. State your induction hypothesis! Without it, the whole proof falls apart. Usually it is just restating your $P(n)$ for some $n \geq 0$ (or, whatever your base case is).

Step 4: Inductive Step. Now consider $P(n+1)$. This is where you try to prove a larger case of the problem than you assumed in your induction hypothesis. What are you trying to prove? Keep this in mind when you do this step. Remember, **use your induction hypothesis somewhere**, and **clearly state where**. If you haven't used your induction hypothesis in this step, then you are not doing a proof by induction. So you'd better need to use it.

Comments

- If your $P(n)$ doesn't mention n in it anywhere, then P isn't a property of n and that's worrisome. P is a function of n, and should be defined in terms of n.
- $P(n)$ is a (boolean) *property*, not a number, so you can*not* manipulate it mathematically, such as $P(n) = 5$, or $P(n+1) < P(n)$.
- Be careful with the base case... sometimes you will need more than one, as with some recurrence relations.
- Recall (or google) the difference between strong and weak induction. They're equivalent in power, but sometimes using one is simpler than using the other.
- Suppose you are trying to prove a particular equation holds, for example, that $x = y$ (for some x and y). It is not mathematically correct to assume the equality is true, then modify both sides of the equation until you get an equation that is actually true. For example,

$$\begin{aligned} x &= y \\ 0 \times x &= 0 \times y \\ 0 &= 0\,. \end{aligned}$$

  This proof "works" for *any* x and y, so obviously this method can get you into trouble. Instead of this method, it is better to start with x, and make modifications to x through a series of steps that somehow ends with y. That is, a proof more like $x = \cdots = y$.

Example

Let $P(n)$ be the statement that $\sum_{i=0}^{n} 2^i = 2^{n+1}-1$. We want to show that $P(n)$ is true for all $n \geq 0$.

Base Case: We want to show that $P(n)$ is true for $n = 0$: that is, that $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$. Fortunately,

$$\begin{aligned} \text{left-hand side (LHS)} &= \sum_{i=0}^{0} 2^i \\ &= 2^0 \\ &= 1 \\ &= 2^1 - 1 \; = \; RHS \; . \end{aligned}$$

Induction Hypothesis: Assume $P(n)$ is true for some $n \geq 0$. That is, assume that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \; .$$

Inductive Step: We want to prove $P(n+1)$, that is, that $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$.

$$\begin{aligned} LHS &= \sum_{i=0}^{n+1} 2^i \\ &= 2^{n+1} + \sum_{i=0}^{n} 2^i \\ &\overset{IH}{=} 2^{n+1} + (2^{n+1} - 1) \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 = RHS \; . \end{aligned}$$

Conclusion: By mathematical induction, we have shown that for all $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.