

# On Distance Coloring

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June 29, 2007

## Abstract

Call a connected undirected graph  $(d, c)$ -colorable if there is a vertex coloring using at most  $c$  colors such that no two vertices of distance  $d$  or less have the same color. It is well known that  $(1, 2)$ -colorability is decidable in linear time, but  $(1, c)$ -colorability for  $c \geq 3$  is *NP*-complete. In [19], Sharp shows that for fixed  $d \geq 2$ , the  $(d, c)$ -colorability problem is solvable in linear time for  $c \leq 3d/2$  and *NP*-complete otherwise. In this note we give an alternative construction that improves the upper time bound as a function of  $d$  for the case  $c \leq 3d/2$ . The construction entails a generalization of the notion of tree decomposition and bounded treewidth [18] to arbitrary overlay graphs, not just trees, which may be of independent interest.

## 1 Introduction

Let  $G = (V, E)$  be a connected undirected graph with no multiple edges or loops. The *distance* between two vertices is the length (number of edges) of a shortest path between them. A  $(d, c)$ -coloring is an assignment of colors to the vertices using at most  $c$  colors such that no two vertices of distance  $d$  or less have the same color. For fixed  $d$  and  $c$ , the  $(d, c)$ -coloring problem is the problem of determining whether a given graph  $G$  has a  $(d, c)$ -coloring and finding one if so.

The classical coloring problem is the special case  $d = 1$ . It is well known that this problem is *NP*-complete for three or more colors [7]. However, for two colors, the problem is decidable in linear time: a graph is colorable with two colors iff it has no odd cycles, which can be determined in linear time by breadth-first search. Thus  $(1, 2)$ -colorability is in *P*, whereas  $(1, c)$ -colorability for  $c \geq 3$  is *NP*-complete.

The  $(d, c)$ -coloring problem is the same as the  $(1, c)$ -coloring problem for the  $d$ th power of the graph, which is obtained simply by adding an edge between any

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two vertices of distance  $d$  or less. However, the  $(d, c)$ -coloring problem may be easier in some cases for small values of  $c$  relative to  $d$ . In [19], the following sharp boundary between easy and hard cases was established: for fixed  $d \geq 2$ ,  $(d, c)$ -coloring is solvable in linear time for  $c \leq 3d/2$  and  $NP$ -complete for  $c > 3d/2$ .

The linear time bound in [19] for the case  $c \leq 3d/2$  is actually only linear in the size of the graph, but exponential in  $d$ . It uses the fact that for  $c \leq 3d/2$ , a graph is  $(d, c)$ -colorable only if it is of bounded treewidth [18]. The algorithm constructs a tree decomposition and uses known algorithms for coloring graphs of bounded treewidth. In [19], a treewidth bound of  $5d$  is obtained. Since the complexity of the algorithm depends so drastically on  $d$ , it is desirable to obtain a bound on treewidth that is as small as possible.

In this paper we give an alternative construction that improves the treewidth bound from  $5d$  to  $2d$ . However, the main interest is not the bound itself, but rather the analysis, which reveals a clear picture of the structure of colorable graphs. Our construction entails a generalization of the notion of tree decomposition and bounded treewidth [18] to allow arbitrary overlay graphs, not just trees, which may be of independent interest. We couple this generalized notion of decomposition with a general construction that obtains such a decomposition from a certain class of edge partitions. One such partition is the one determined by the biconnected components of the graph, which gives a tree decomposition, but there are others. In particular, the partition we use gives a bounded-width cycle decomposition of  $G$ .

Finally, our analysis sheds light on a particular source of  $NP$ -hardness in graph problems, namely the ability to propagate information *nonlinearly* in the graph. To understand what we mean by this, recall that typical  $NP$ -hardness proofs in graphs involve the construction of certain “widgets” that can propagate information in the graph in a controlled way. These widgets can be quite intricate. For example, to show that planar 3-colorability is  $NP$ -complete [7], one can use the *crossover widget* shown in Fig. 1. This widget has the properties (i) it is planar, (ii) any legal 3-coloring must color the opposite corners the same, and (iii) every 3-coloring of the corners with opposite corners the same extends to a legal 3-coloring of the whole widget. Thus, by replacing edge crossings with the widget, 3-colorability of arbitrary graphs can be reduced to 3-colorability of planar graphs. We can think of the widget in Fig. 1 as propagating coloring information from the east corner to the west corner and simultaneously from the north corner to the south corner.

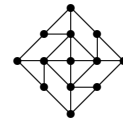


Fig. 1

However, to establish  $NP$ -hardness, it is not enough just to be able to propagate information in the graph; it must also be possible to *duplicate* information so that it can be propagated *nonlinearly*. Here we are using the term *linear* in the same sense as linear logic [8]. In linear logic, proofs may not freely duplicate assumptions and must consume all occurrences of all assumptions. Every assertion that is produced is consumed exactly once. Thus the proof rules of linear logic do not allow the nonlinear propagation of information.

A similar phenomenon arises in combinatorial constraint problems. For ex-

ample, in the Boolean satisfiability problem, a 3CNF Boolean formula may contain several occurrences of the same variable. Information is propagated by the constraint that all occurrences of that variable must have the same truth value. The general 3CNF satisfiability problem is *NP*-complete, but the problem is efficiently solvable under the restriction that each variable occur at most twice. Intuitively, this restriction prevents the nonlinear propagation of information.

Likewise, in the distance- $d$  coloring problem, the restriction to  $3d/2$  or fewer colors essentially prevents the nonlinear propagation of information. A



Fig. 2

paradigmatic example is the graph illustrated in Fig. 2 with  $d = 6$ . This graph is  $(6, 10)$ -colorable, and any legal coloring imposes certain constraints on the colors of any node connected to one of the three terminal nodes. In the  $(6, 10)$ -coloring problem, this graph can be used as a widget to propagate coloring constraints nonlinearly. Consequently,  $(6, 10)$ -coloring is *NP*-complete [19].

On the other hand, the graph of Fig. 2 is not  $(6, 9)$ -colorable, so it is a  $(6, 9)$ -*forbidden subgraph* in the sense that it cannot appear as a subgraph of any  $(6, 9)$ -colorable graph. This severely restricts the form of  $(6, 9)$ -colorable graphs to the extent that they are efficiently recognizable and efficiently colorable.

This paper is organized as follows. In Section 2, we introduce a generalization of the notion of tree decomposition [18]. Also in that section, we indicate how such decompositions can be obtained from a certain class of edge partitions, of which biconnected components are a special case. In Section 3, we show that for  $c \leq 3d/2$ ,  $(d, c)$ -colorable graphs have a tree decomposition of width at most  $2d$ , and we can find such a decomposition efficiently. In one case, this involves finding a cycle decomposition of width at most  $d$  and stringing the two strands of the cycle together. We conclude in Section 4 with some observations regarding the homology of colorable graphs.

## 2 Decomposition

In this section we introduce a generalization of the notion of tree decomposition [18]. In a tree decomposition, the overlay graph is a tree. In our generalization, the overlay graph may be any graph.

### 2.1 Definition of Decomposition

Let  $i, j, k$  be vertices of an undirected graph  $G = (V, E)$ . The set  $[i, k]$  is the set of all vertices lying on all shortest paths from  $i$  to  $k$ , inclusive, in  $G$ . The set  $[i, k]$  is called an *interval*. We say that  $j$  is *between  $i$  and  $k$*  if  $j \in [i, k]$ . A set  $A$  of vertices is *convex* if  $[i, k] \subseteq A$  whenever  $i, k \in A$ . Some basic properties of intervals and convexity in graphs are developed in [14, 15, 16].

**Definition 2.1** A *decomposition* of  $G$  is a triple  $(T, F, X)$  consisting of an undirected *overlay graph*  $(T, F)$  and a map  $X : T \rightarrow 2^V$  associating a subset  $X_i \subseteq V$  with each  $i \in T$  satisfying the following properties.

- (i) The  $X_i$  cover  $V$ ; that is,  $V = \bigcup_{i \in T} X_i$ .
- (ii) For all edges  $(s, t) \in E$ , there exists  $i \in T$  such that  $s, t \in X_i$ .
- (iii) If  $j$  is between  $i$  and  $k$  in  $(T, F)$ , then  $X_i \cap X_k \subseteq X_j$ .

Equivalent to (iii) is the property:

- (iv) For any vertex  $u$ , the set  $\{i \mid u \in X_i\}$  is convex.

The *width* of  $(T, F, X)$  is  $\max_i |X_i|$ . □

A *tree decomposition* is just a decomposition in which the overlay graph is an undirected tree. In this case, the intervals are just unique simple paths between pairs of vertices and the convex sets are subtrees. We will also consider *cycle decompositions* and *line decompositions* in which the overlay graphs are undirected cycles and paths, respectively.

## 2.2 Decompositions from Edge Partitions

Decompositions can be obtained from certain edge partitions. Let  $P \subseteq 2^E$  be an edge partition of  $G = (V, E)$ ; that is, a collection of pairwise disjoint nonempty sets of edges whose union is  $E$ . For  $A \in P$ , let  $V(A) \subseteq V$  be the set of endpoints of edges in  $A$ . Suppose  $P$  satisfies the following additional property:

**Condition 2.2** *If  $A, B \in P$  and  $A \neq B$ , then  $|V(A) \cap V(B)| \leq 1$ .*

Thus no two partition elements share more than one vertex. If  $P$  satisfies this condition, then we call any vertex common to two or more partition elements a *local articulation point*.

Such partitions give rise to decompositions of  $G$  as follows:

**Lemma 2.3** *Let  $P$  be an edge partition of  $G$  satisfying Condition 2.2. Let*

$$\begin{aligned} T &= P \cup \{u \mid V(A) \cap V(B) = \{u\} \text{ for some } A, B \in P\} \\ F &= \{(u, A) \mid u \in V(A)\} \\ X_A &= V(A) \\ X_u &= \{u\}. \end{aligned}$$

*The structure  $(T, F, X)$  is a decomposition of  $G$ .*

*Proof.* Properties (i) and (ii) of Definition 2.1 follow from the fact that  $G$  is connected and  $P$  is a partition of the edges. For property (iii),  $X_u$  and  $X_v$  cannot intersect for  $u \neq v$ . If  $X_u$  and  $X_A$  intersect, then  $(u, A) \in F$ , and there is no other vertex of  $(T, F)$  between  $u$  and  $A$ , therefore (iii) holds vacuously. Finally, if  $X_A$  and  $X_B$  intersect,  $A \neq B$ , then by Condition 2.2,  $X_A \cap X_B = \{u\}$  for some vertex  $u$ , thus  $[A, B] = \{A, u, B\}$ , and  $X_A \cap X_B = \{u\} = X_u$ . □

A special case of such a decomposition is given by the biconnected components of  $G$  and their articulation points [10]. Recall that the biconnected components are the equivalence classes of edges with respect to the equivalence relation defined by:  $e \sim e'$  if  $e$  and  $e'$  lie on a common simple cycle. In this case, the decomposition provided by Lemma 2.3 is a tree decomposition of  $G$ , and the width is the maximum size of a biconnected component. In this decomposition, the sets  $X_A$  for  $A \in P$  are the biconnected components of  $G$  and the sets  $X_u$  correspond to the articulation points. We will see a generalization of this construction in Section 3.2.

### 3 An Algorithm

Let  $c, d \geq 2$  be fixed constants,  $c \leq 3d/2$ . Henceforth, *colorable* means  $(d, c)$ -colorable. Let  $G = (V, E)$  be a connected undirected graph. Let  $d(\cdot, \cdot)$  be the shortest-path distance function in  $G$ .

In this section we describe an algorithm to either construct a constant-width tree, line, or cycle decomposition of  $G$ , which can then be used to give a tree decomposition of  $G$  of width at most  $2d$ , or declare that  $G$  is not colorable. This algorithm can be used in conjunction with known algorithms for coloring graphs of bounded treewidth [12, 17, 3] to either give a coloring of  $G$  or declare that no such coloring exists. The algorithm is linear in the size of  $G$  for fixed  $d$ .

The algorithm proceeds in several steps, which we treat in the indicated subsections.

§3.1 We first partition the graph into biconnected components. We show that there is at most one large component  $G'$ , otherwise  $G$  is not colorable. If there are no large components, we are done. Otherwise, the large component contains a simple cycle  $T$  of length at least  $2d + 2$ , which can be found in linear time. The cycle  $T$  is called the *trunk*.

§3.2 We construct a cycle decomposition of  $G'$  by defining an edge partition satisfying Condition 2.2.

§3.3 Adding the biconnected components removed in §3.1 back in, we show that we obtain a cycle decomposition of width at most  $c - d/2$ .

§3.4 Finally, we show how to obtain a line decomposition of width  $2w$  from a cycle decomposition of width  $w$  by collapsing the two strands of the cycle.

Any one of these steps may fail if  $G$  is not colorable, but this will occur in a recognizable way.

#### 3.1 Biconnected Components

We first find the biconnected components of  $G$ . This can be done in linear time [10]. As observed, the biconnected components and their articulation points provide a tree decomposition of  $G$ . A biconnected component is *small* if its diameter is at most  $d/2$ , otherwise it is *large*.

**Lemma 3.1** *If  $G$  is colorable, then there is at most one large component.*

*Proof.* If there were two large components, then each would have a cycle of length at least  $d$ . Since the graph is connected, the two large components would be connected by an isthmus, and one could construct a forbidden subgraph similar to the one shown in Fig. 2.  $\square$

A large component  $G'$  must be biconnected and its diameter must be at least  $\lfloor d/2 \rfloor + 1$ . If  $3d/4 < \text{diam } G' \leq d$ , then  $G'$  (hence  $G$ ) is not colorable, since there would be a simple cycle of length greater than  $3d/2$  in a set of diameter  $d$ . So either  $\text{diam } G' \leq 3d/4$  or  $\text{diam } G' > d$ . If the former, or if there is no large component at all, then each biconnected component is of size at most  $c$ , otherwise the graph is not colorable. In this case we have a tree decomposition of width at most  $c \leq 3d/2$  consisting of the biconnected components, and we are done.

So assume  $\text{diam } G' \geq d + 1$ . There must exist nodes  $a, b$  such that  $d(a, b) \geq d + 1$ , and since  $G'$  is biconnected, there is a simple cycle  $T$  of length at least  $2d + 2$  containing  $a$  and  $b$ . We will call the cycle  $T$  the *trunk*. The nodes  $a, b$  and the trunk can be found in linear time by depth-first or breadth-first search.

### 3.2 A Cycle Decomposition

In this section we assume that we have identified a biconnected component  $G'$  of diameter at least  $d + 1$ . This implies the existence of a simple cycle  $T$  in  $G'$  of length at least  $2d + 2$  and two nodes  $a$  and  $b$  on  $T$  of distance at least  $d + 1$ . We will show how to find a cycle decomposition of  $G'$  of small width, provided  $G'$  is colorable. The picture to keep in mind is Fig. 3.

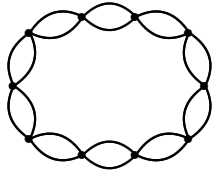


Fig. 3

First we prove a lemma that will allow us to construct a suitable edge partition consisting of the equivalence classes of a certain equivalence relation  $\stackrel{d}{\sim}$ . This partition is very much like biconnected components, except that we impose a bound of  $d$  on the length of cycles. The following technical lemma is needed for transitivity.

**Lemma 3.2** *If  $G'$  is colorable, then  $G'$  has no simple cycle of length  $k$  for any  $d + 1 \leq k \leq 2d + 1$ .*

*Proof.* Suppose for a contradiction that there were a simple cycle  $K$  of length  $k$  with  $d + 1 \leq k \leq 2d + 1$ . Then  $\text{diam } K = \lfloor k/2 \rfloor \leq d$ , so by colorability we must have  $k \leq c \leq 3d/2$ .

The cycles  $T$  and  $K$  must have a node in common; if not, then since the graph is connected,  $T$  and  $K$  would be connected by an isthmus, and we could construct a forbidden subgraph like the one shown in Fig. 2.

Since  $k \leq 3d/2$ , there must exist at least  $|T| - k \geq 2d + 2 - \lfloor 3d/2 \rfloor = \lfloor d/2 \rfloor + 2$  nodes in  $T \setminus K$ . Let  $x$  be an arbitrary node in  $T \setminus K$ . Let  $d(x, K)$  denote the distance from  $x$  to the closest element of  $K$ . Since  $T$  and  $K$  share at least one node,  $x$  is contained in a maximal segment  $P$  of  $T$  disjoint from  $K$ . The

segment  $P$  may have no more than  $\lfloor d/2 \rfloor - 1$  nodes, otherwise we could construct a forbidden subgraph like the one shown in Fig. 2 consisting of nodes from  $P$  and nodes from  $K$ . Therefore

$$d(x, K) \leq \lceil (\lfloor d/2 \rfloor - 1)/2 \rceil. \quad (1)$$

We claim now that

$$d(a, K) > \lceil (\lfloor 3d/2 \rfloor - k)/2 \rceil \geq 0. \quad (2)$$

If not, then

$$\begin{aligned} d(a, b) &\leq d(a, K) + \lfloor k/2 \rfloor + d(K, b) \\ &\leq \lceil \lfloor 3d/2 \rfloor / 2 - k/2 \rceil + \lfloor k/2 \rfloor + \lceil (\lfloor d/2 \rfloor - 1)/2 \rceil \\ &\leq \lceil \lfloor 3d/2 \rfloor / 2 - \lfloor k/2 \rfloor \rceil + \lfloor k/2 \rfloor + \lfloor \lfloor d/2 \rfloor / 2 \rfloor \\ &= \lceil \lfloor 3d/2 \rfloor / 2 \rceil + \lfloor \lfloor d/2 \rfloor / 2 \rfloor \\ &= \lceil d/2 + \lfloor d/2 \rfloor / 2 \rceil + \lfloor \lfloor d/2 \rfloor / 2 \rfloor \\ &\leq \lceil d/2 \rceil + \lceil \lfloor d/2 \rfloor / 2 \rceil + \lfloor \lfloor d/2 \rfloor / 2 \rfloor \\ &= \lceil d/2 \rceil + \lfloor d/2 \rfloor \\ &= d, \end{aligned}$$

contradicting the assumption that  $d(a, b) \geq d + 1$ . By symmetry, (2) also gives a lower bound on  $d(b, K)$ .

Since every node  $x \in T$  is of distance at most  $\lceil (\lfloor d/2 \rfloor - 1)/2 \rceil$  from  $K$ , and since every pair of nodes on  $K$  are of distance at most  $\lfloor k/2 \rfloor \leq \lfloor \lfloor 3d/2 \rfloor / 2 \rfloor$ , the distance between any element of  $T$  and any element of  $K$  is bounded by

$$\begin{aligned} \lfloor \lfloor 3d/2 \rfloor / 2 \rfloor + \lceil (\lfloor d/2 \rfloor - 1)/2 \rceil &\leq \lfloor \lfloor 3d/2 \rfloor / 2 \rfloor + \lfloor \lfloor d/2 \rfloor / 2 \rfloor \\ &\leq 3d/4 + d/4 \\ &= d. \end{aligned} \quad (3)$$

By (2),  $a \notin K$ . Let  $Q$  be the maximal segment of  $T$  containing  $a$  and disjoint from  $K$ . By (3),  $\text{diam}(Q \cup K) \leq d$ . But  $Q$  contains at least  $2d(a, K) - 1$  elements of  $T \setminus K$ , thus

$$\begin{aligned} |Q \cup K| &\geq 2d(a, K) - 1 + k \\ &\geq 2(\lceil (\lfloor 3d/2 \rfloor - k)/2 \rceil + 1) - 1 + k \\ &= 2\lceil (\lfloor 3d/2 \rfloor - k)/2 \rceil + 1 + k \\ &\geq 2((\lfloor 3d/2 \rfloor - k)/2) + 1 + k \\ &= \lfloor 3d/2 \rfloor + 1, \end{aligned}$$

contradicting colorability.  $\square$

For edges  $e, e'$  in  $G'$ , define  $e \stackrel{d}{\sim} e'$  if  $e$  and  $e'$  lie on a simple cycle of length at most  $d$ . This definition is identical to the definition of the edge partition for biconnected components except for the length restriction. In the presence of the length restriction, the only difficulty would be transitivity, but this is taken care of by Lemma 3.2.

**Lemma 3.3** *If  $G'$  is colorable, then the relation  $\overset{d}{\sim}$  is an equivalence relation.*

*Proof.* Reflexivity  $e \overset{d}{\sim} e$  follows from the fact that the edge  $e$  and its two endpoints constitute a simple cycle. The relation is symmetric, since  $e$  and  $e'$  can be interchanged in the definition of  $\overset{d}{\sim}$ .

For transitivity, suppose  $(u, v) \overset{d}{\sim} (u', v')$  and  $(u', v') \overset{d}{\sim} (u'', v'')$ . Let  $C$  and  $C'$  be two simple cycles of length at most  $d$  such that  $(u, v)$  and  $(u', v')$  occur on  $C$  and  $(u', v')$  and  $(u'', v'')$  occur on  $C'$ . Assume  $u, u', v', v$  occur in that order around  $C$ . Let  $x$  be the first vertex on the segment of  $C$  from  $u$  to  $u'$  that also lies in  $C'$ ;  $x$  must exist, since  $u' \in C'$ . Let  $y$  be the first vertex on the segment of  $C$  from  $v$  to  $v'$  that also lies in  $C'$ ;  $y$  must exist, since  $v' \in C'$ . Also,  $x \neq y$ , since  $C$  is simple. Let  $P$  be the segment of  $C$  between  $x$  and  $y$  containing  $(u, v)$  and let  $P'$  be the segment of  $C'$  between  $x$  and  $y$  containing  $(u'', v'')$ . Then  $P$  and  $P'$  intersect only in  $x$  and  $y$ , and together they form a simple cycle  $K$  containing  $(u, v)$  and  $(u'', v'')$ . The length of  $K$  is at most  $|C| + |C'| - 2 \leq 2d - 2$ , so by Lemma 3.2, its length is at most  $d$ , therefore  $(u, v) \overset{d}{\sim} (u'', v'')$ .  $\square$

**Lemma 3.4** *If  $G'$  is colorable, then no two  $\overset{d}{\sim}$ -equivalence classes have more than one node in common.*

*Proof.* If  $u$  and  $v$  were both nodes of distinct  $\overset{d}{\sim}$ -classes  $X$  and  $Y$ , then  $u$  and  $v$  would be of distance at most  $d/2$  in  $X$  and at most  $d/2$  in  $Y$ . Combining shortest paths in  $X$  and  $Y$  between  $u$  and  $v$  would yield a simple cycle of length at most  $d$  with edges from both  $X$  and  $Y$ , thus  $X = Y$ . This is a contradiction.  $\square$

Lemmas 3.3 and 3.4 allow us to apply the theory of Section 2. Condition 2.2 follows from Lemma 3.4. Thus the  $\overset{d}{\sim}$ -equivalence classes give a decomposition of  $G'$  of width at most  $c$ . Later, in Section 3.3, we will improve the width bound.

To show that the decomposition is a cycle decomposition, we need to show how to identify a local articulation point on  $T$ . For  $s, t \in T$ , let  $(s, t)$  denote the open interval  $[s, t] \setminus \{s, t\}$ . A *chord* is a path between two nodes of  $T$  none of whose intermediate nodes lie on  $T$ . We use the notation  $\langle s, t \rangle$  to represent a chord with endpoints  $s, t \in T$ . A node  $u \in T$  is *subtended* by a chord  $\langle s, t \rangle$  if  $u$  lies strictly between  $s$  and  $t$  on a shortest path between  $s$  and  $t$  in  $T$ ; that is, if  $s \neq u \neq t$  and  $u \in [s, t]$ . Recall that  $[s, t]$  is the set of all nodes on all shortest paths between  $s$  and  $t$  in  $T$ . There is only one such path unless  $T$  is of even length and  $s$  and  $t$  are diametrically opposed. The chord  $\langle s, t \rangle$  subtends all the nodes in  $(s, t)$ .

**Lemma 3.5** *If  $G'$  is colorable, then there exists a local articulation point on  $T$ . No local articulation point is subtended by a chord.*

*Proof.* If a node  $x$  on  $T$  is subtended by a chord, then the two edges on  $T$  adjacent to  $x$  are  $\overset{d}{\sim}$ -equivalent, because the length of the chord and the length

of the interval of  $T$  subtended are at most  $d/2$ , otherwise we could construct a forbidden subgraph like the one shown in Fig. 2. Thus if every node of  $T$  is subtended by a chord, then by transitivity, all edges of  $T$  are  $\overset{d}{\sim}$ -equivalent. But this cannot be, because no two nodes of distance greater than  $d/2$  on the same segment of  $T$  between  $a$  and  $b$  can both be in  $V(A)$  for some  $\overset{d}{\sim}$ -class  $A$ , because the diameter of  $A$  is at most  $d/2$ , and this would short-circuit the distance from  $a$  to  $b$ .  $\square$

In fact, by the same argument, the trunk contains at least four local articulation points, thus contains edges from at least four  $\overset{d}{\sim}$ -classes. Starting from  $a$  and moving around the trunk, there must be a local articulation point  $x$  within distance  $d/2$  of  $a$ , and there must be a second local articulation point  $y$  within distance  $d/2$  of  $x$ , both of which must be encountered before reaching  $b$ . There must be at least two other local articulation points on the other path on  $T$  from  $a$  to  $b$ .

**Lemma 3.6** *If  $G'$  is colorable, then  $G'$  consists of a cycle of biconnected subgraphs, each of diameter at most  $d/2$ . Each biconnected subgraph is connected to its two neighbors on the cycle by a local articulation point. We can find the decomposition in linear time.*

*Proof.* The existence is clear from Lemma 3.5. To find the decomposition, first find a local articulation point  $s$ . This can be done in linear time, since every set of size  $\lfloor d/2 \rfloor$  must contain a point of  $T$ , and there is a local articulation point no more than  $\lfloor d/2 \rfloor$  from any node of  $T$ . A local articulation point is recognizable by breadth-first search from  $s$  down to a depth of  $d/2$ , since it is not subtended by a chord. We break the graph at  $s$ , forming two copies of  $s$ , then find the biconnected components of the resulting graph, which gives the desired decomposition.  $\square$

### 3.3 Bounding Cycle Width

Let  $C$  be the set of small biconnected components of  $G$  that we removed in Section 3.1, and let  $P$  be the set of partition elements of  $G'$  constructed in Section 3.2. In this section we show that, even in the presence of the components  $C$ , we can obtain a cycle decomposition of width at most  $c - d/2$ .

Let  $N = V \setminus T$  the set of non-trunk nodes of  $G$ . Except for articulation points,  $N$  contains all nodes of  $V(A)$  for any small biconnected component  $A$  of  $G$ .

Let  $I$  be an interval in  $T$  of length exactly  $\lfloor d/2 \rfloor$ , which contains exactly  $\lfloor d/2 \rfloor + 1$  nodes. Note that  $T$  extends at least a distance  $\lfloor d/2 \rfloor$  on both sides of  $I$ . If  $U \subseteq V$ , a  $U$ -path is a path all of whose intermediate (non-endpoint) nodes are in  $U$ . Let  $B$  be the set of nodes in  $N$  connected to a node in  $I$  via an  $N$ -path.

If the interval  $I$  contains two local articulation points bounding an element of  $P$ , then  $B \cup I$  contains that component of  $P$  and all small biconnected components of  $G$  connected to it via an  $N$ -path. Moreover, every small biconnected

component of  $G$  is so connected to some component of  $P$ , and every component of  $P$  is so bounded by some  $I$ . We wish to show

**Lemma 3.7** *If  $G$  is  $(d, c)$ -colorable, then  $|B \cup I| \leq c - d/2$ .*

This will follow from a sequences of lemmas. First we show how to simplify the graph without loss of generality. Delete any edge between a node in  $N$  and a node in  $T \setminus I$ ; every node in  $B$  is still connected to  $I$  via an  $N$ -path, so the definition of  $B$  after deleting these edges would still give the same set. Moreover, the graph is still  $(d, c)$ -colorable, since deleting edges cannot make it less colorable. Find a spanning tree of the remaining graph containing all edges of  $I$ . One edge of  $T$  was removed, but add it back in. The resulting graph is still colorable, and all nodes of  $B$  are still connected to  $I$  via an  $N$ -path. Delete all nodes not connected to  $I$ , and redefine  $G$  to be the resulting graph.

Our simplifications ensure that the graph  $G$  consists of the trunk  $T$  with a disjoint set of trees branching out from nodes of  $I$ . Let  $B_u$  be the tree of nodes in  $B$  connected to  $u \in I$  by an  $N$ -path. The sets  $B_u$  are disjoint and span  $B$ . For every node  $x \in B$ , there is a unique  $u \in I$  such that  $x \in B_u$ , and for  $u \neq v$ ,  $B_u$  and  $B_v$  are connected only through  $u$  and  $v$ .

**Lemma 3.8** *Let  $u \in I$ . Then  $d(x, u) \leq d/2 - 1$  for all  $x \in B_u$ .*

*Proof.* If not, then there exists  $x \in B_u$  with  $d(x, u) = \lfloor d/2 \rfloor$ . The set of elements on a shortest path in  $B_u$  from  $x$  to  $u$  along with an interval of  $d + 1$  elements of  $T$  of which  $u$  is the median element form a set of size  $\lfloor 3d/2 \rfloor + 1$  and diameter  $d$ , contradicting colorability.  $\square$

**Lemma 3.9**  $\text{diam}(B \cup I) \leq d - 1$ .

*Proof.* If  $\text{diam}(B \cup I) \geq d$ , then there exist  $x, y \in B \cup I$  such that  $d(x, y) = d$ . Let  $u, v \in I$  such that  $x \in B_u \cup \{u\}$  and  $y \in B_v \cup \{v\}$ . By Lemma 3.8,  $u \neq v$ . Assume without loss of generality that  $u$  is to the left of  $v$ . Let  $A$  consist of all nodes on a shortest path from  $x$  to  $y$  (which includes a shortest  $B_u$ -path from  $x$  to  $u$ , followed by a shortest  $T$ -path from  $u$  to  $v$ , followed by a shortest  $B_v$ -path from  $v$  to  $y$ ) along with  $d(x, u)$  nodes of  $T$  immediately to the left of  $u$  and  $d(y, v)$  nodes of  $T$  immediately to the right of  $v$ . Then  $\text{diam} A = d$  and

$$\begin{aligned} |A| &= 2d(x, u) + 2d(y, v) + d(u, v) + 1 \\ &= 2(d(x, u) + d(u, v) + d(v, y)) - d(u, v) + 1 \\ &= 2d(x, y) - d(u, v) + 1 \\ &\geq 2d - d/2 + 1 \\ &= 3d/2 + 1, \end{aligned}$$

contradicting colorability.  $\square$

**Lemma 3.10** *Let  $z \in I$  be chosen so as to minimize the maximum distance to any other node in  $B \cup I$ . Then  $d(z, x) \leq d/2$  for all  $x \in B \cup I$ .*

*Proof.* Let  $x \in B \cup I$  maximizing  $d(z, x)$ . If  $x \in B_z$ , then we are done by Lemma 3.8. Otherwise, assume without loss of generality that  $x \in B_u$  for some  $u$  to the left of  $z$  (the argument is symmetric if  $u$  is to the right of  $z$ ). Let  $y$  be such that  $d(z, y)$  is maximum over all elements of all  $B_v$  for  $v = z$  or  $v$  to the right of  $z$ . Then

$$d(x, y) = d(x, z) + d(z, y), \quad (4)$$

since the shortest path from  $x$  to  $y$  goes through  $z$ , and

$$d(x, z) \leq d(z, y) + 1 \quad (5)$$

by choice of  $z$ : if  $d(x, z) \geq d(z, y) + 2$ , then the node immediately to the left of  $z$  would have been a better choice. Using (4), (5), and Lemma 3.9,

$$d - 1 \geq d(x, y) = d(x, z) + d(z, y) \geq 2d(x, z) - 1,$$

therefore  $d(x, z) \leq d/2$ . □

**Lemma 3.11**  $|B| \leq c - d - 1$ .

*Proof.* Let  $D$  consist of  $B$  along with an interval of  $d + 1$  elements of  $T$  with the  $z$  of Lemma 3.10 as median element. By Lemma 3.10,  $\text{diam } D = d$ , thus  $|D| \leq c$  by colorability, therefore  $|B| = |D| - (d + 1) \leq c - d - 1$ . □

Lemma 3.7 now follows, since

$$|B \cup I| = |B| + |I| \leq c - d - 1 + d/2 + 1 = c - d/2.$$

### 3.4 Converting Cycles to Lines

We have shown that  $G$  has a cycle decomposition of width at most  $c - d/2$ . Our main result will follow immediately from the following lemma.

**Lemma 3.12** *If  $G$  has a cycle decomposition of width  $w$ , then  $G$  has a line decomposition of width at most  $2w$ .*

*Proof.* Intuitively, grasp the cycle by diametrically opposed nodes and pull, creating a line with two strands.

Formally, let  $(T, F, X)$  be a cycle decomposition of  $G = (V, E)$  of width  $w$ . Suppose  $(T, F)$  is a cycle of length  $m + 1$  with the vertices of  $T$  named  $0, 1, 2, \dots, m$  in order around the cycle. Consider the line  $(T', F', X')$  with

$$\begin{aligned} T' &= \{0, 1, 2, \dots, \lfloor m/2 \rfloor\}, \\ F' &= \{(i, i + 1) \mid 0 \leq i \leq \lfloor m/2 \rfloor - 1\}, \\ X'_i &= X_i \cup X_{m-i}. \end{aligned}$$

We wish to show that this is a line decomposition of  $G$  of width at most  $2w$ . Properties (i) and (ii) of line decompositions hold trivially, and it is clearly of width at most  $2w$ , thus it remains to show property (iii).

Suppose  $0 \leq i < j < k \leq \lfloor m/2 \rfloor$ . We wish to show  $X'_i \cap X'_k \subseteq X'_j$ , or

$$(X_i \cup X_{m-i}) \cap (X_k \cup X_{m-k}) \subseteq X_j \cup X_{m-j},$$

which by distributivity reduces to the following four inclusions:

$$X_i \cap X_k \subseteq X_j \cup X_{m-j} \quad (6)$$

$$X_i \cap X_{m-k} \subseteq X_j \cup X_{m-j} \quad (7)$$

$$X_{m-i} \cap X_k \subseteq X_j \cup X_{m-j} \quad (8)$$

$$X_{m-i} \cap X_{m-k} \subseteq X_j \cup X_{m-j}. \quad (9)$$

These inclusions hold by virtue of the fact that  $(T, F, X)$  is a cycle decomposition. Properties (6) and (9) are immediate, since  $j \in [i, k]$ , and by symmetry  $m-j \in [m-i, m-k]$ , therefore  $X_i \cap X_k \subseteq X_j$  and  $X_{m-i} \cap X_{m-k} \subseteq X_{m-j}$ .

For (7) and (8), there are two cases, depending on whether

$$(m-k) - i \bmod m + 1 \leq \lfloor (m+1)/2 \rfloor \text{ or}$$

$$i - (m-k) \bmod m + 1 \leq \lfloor (m+1)/2 \rfloor.$$

In the former case,  $j \in [i, k]$  and  $k \in [i, m-k]$ , therefore  $j \in [i, m-k]$ , and by symmetry,  $m-j \in [m-i, k]$ . These imply that  $X_i \cap X_{m-k} \subseteq X_j$  and  $X_{m-i} \cap X_k \subseteq X_{m-j}$ .

In the latter case, we have  $k - (m-i) \bmod m + 1 \leq \lfloor (m+1)/2 \rfloor$ . Then  $j \in [i, k]$  and  $i \in [m-i, k]$ , therefore  $j \in [m-i, k]$ , and by symmetry,  $m-j \in [i, m-k]$ . These imply that  $X_{m-i} \cap X_k \subseteq X_j$  and  $X_i \cap X_{m-k} \subseteq X_{m-j}$ .  $\square$

## 4 Homology of Colorable Graphs

Colorability has deeper implications regarding the cycle structure of the graph that are best expressed in terms of homology. The main observation is Theorem 4.2 below, which says that modulo short simple cycles, there is at most one long simple cycle in the graph.

Let  $G = (V, E)$  be a connected undirected graph. Let  $\vec{E}$  be the set of *directed* edges of  $G$ , consisting of two directed edges  $uv$  and  $vu$  for each undirected edge with endpoints  $u, v$  of  $G$ . Let  $\Delta$  be the set of oriented simple cycles. We write  $\langle u_1 u_2 \cdots u_n \rangle$  for the oriented cycle that goes through  $u_1, u_2, \dots, u_n, u_1$  in that order. Call an element of  $\Delta$  *short* if it is of length at most  $2d+1$ , otherwise *long*. Let  $\Delta^d$  be the set of short oriented simple cycles. Let  $H_0, H_1$ , and  $H_2$  be the free abelian groups on generators  $V, \vec{E}$ , and  $\Delta$ , respectively. Let  $H_2^d$  be the subgroup of  $H_2$  generated by  $\Delta^d$ .

The free abelian group  $H$  on a finite set of generators  $X$  can be regarded as the set of formal sums  $\sum_{e \in X} k_e e$ , where the  $k_e \in \mathbb{Z}$  are integer coefficients. This also forms a  $\mathbb{Z}$ -module of dimension  $|X|$  with standard basis  $X$ . We have a natural inner product defined by  $e \cdot e' = 1$  if  $e = e'$  and 0 if  $e \neq e'$  for basis elements  $e, e' \in X$ , extended uniquely to  $H^2 \rightarrow \mathbb{Z}$  by bilinearity. For a basis element  $e \in X$ , the projection  $x \mapsto e \cdot x$  gives the coefficient of  $e$  in the

expression  $x$ . We apply this construction in  $H_2$ ,  $H_1$  and  $H_0$  with standard bases  $\Delta$ ,  $\vec{E}$ , and  $V$ , respectively.

For an object  $\alpha$  in  $\Delta$  or  $\vec{E}$ , we denote by  $\bar{\alpha}$  the object with the opposite orientation. For example, if  $\alpha$  is the oriented simple cycle  $\langle uvw \rangle$ , then  $\bar{\alpha} = \langle wvu \rangle$ , and if  $e$  is the directed edge  $uv$ , then  $\bar{e} = vu$ . The maps  $\bar{\cdot} : \Delta \rightarrow \Delta$  and  $\bar{\cdot} : \vec{E} \rightarrow \vec{E}$  extend by linearity to group homomorphisms  $\bar{\cdot} : H_2 \rightarrow H_2$  and  $\bar{\cdot} : H_1 \rightarrow H_1$ , respectively.

Let  $\partial_1 : H_2 \rightarrow H_1$  and  $\partial_0 : H_1 \rightarrow H_0$  be the *boundary homomorphisms* defined by: for an oriented simple cycle  $\alpha$ ,  $\partial_1(\alpha)$  is the sum of the directed edges in  $\alpha$ , and for a directed edge  $uv$ ,  $\partial_0(uv) = v - u$ . For example, if  $\alpha$  is the oriented simple cycle  $\langle uvw \rangle$ , then  $\partial_1(\alpha) = uv + vw + wu$ .

If  $e$  is a directed or undirected edge with endpoints  $u, v$ , let  $t_e \in \Delta$  be the unique cycle  $\langle uv \rangle$  of length 2 consisting of the two directed edges  $uv$  and  $vu$ . For  $e \in \vec{E}$ ,  $\partial_1(t_e) = e + \bar{e}$  and  $t_e = t_{\bar{e}}$ . Let  $H_2^2$  be the subgroup of  $H_2$  generated by the  $t_e$ .

The *cycle group*  $C \subseteq H_1$  is the kernel of  $\partial_0$ , which is the same as the image of  $H_2$  under  $\partial_1$  (Lemma 4.1(i)). Let  $C^d$  be the image of  $H_2^d$  under  $\partial_1$ . The group  $C^d$  is the subgroup of the cycle group generated by (the directed edges of) the short oriented simple cycles.

Let  $P_i$  be the positive orthant of  $H_i$ ,  $0 \leq i \leq 2$ . This is the set of expressions with only nonnegative coefficients. The set  $P_i$  induces a partial order  $\leq$  on  $H_i$  defined by:  $\alpha \leq \beta$  iff  $\beta - \alpha \in P_i$ . An element is called *positive* if it is contained in  $P_i$ .

**Lemma 4.1**

- (i)  $\partial_1(H_2) = C$ .
- (ii)  $\partial_1(P_2) = P_1 \cap C$ .

*Proof.* For the forward inclusion of (i), a standard argument shows the image  $\partial_1(H_2)$  is contained in  $C = \ker \partial_0$ ; that is,  $\partial_0 \circ \partial_1 = 0$ . For example, if  $\alpha = \langle uvw \rangle$ , then

$$\partial_0(\partial_1(\alpha)) = \partial_0(uv + vw + wu) = (v - u) + (w - v) + (u - w) = 0.$$

The forward inclusion of (ii) follows from the observation that the image of a positive element under  $\partial_1$  is positive in  $H_1$ .

For the reverse inclusion, we prove (ii) first. This argument is similar to the proof of Euler's theorem that a cycle in an undirected graph can be written as a union of simple cycles. If  $x \in P_1$  and  $v \in V$ , define the *indegree* of  $v$  in  $x$  to be the sum of the coefficients in  $x$  of all edges entering  $v$ , and define the *outdegree* of  $v$  in  $x$  to be the sum of the coefficients in  $x$  of all edges exiting  $v$ . Observe that for positive  $x$ , membership in  $\ker \partial_0$  is equivalent to the condition that the indegree of every vertex is equal to its outdegree; equivalently,  $v \bullet \partial_0(x) = 0$  for all  $v$ .

Let  $x \in P_1 \cap \ker \partial_0$ . Starting at any node of nonzero outdegree, trace a cycle  $\alpha_1$ , following edges with positive coefficients in  $x$  until encountering a

node previously seen. It is always possible to continue, since the outdegree of any node equals the indegree. Subtract off the edges of  $\alpha_1$ ; the resulting expression  $x - \partial_1(\alpha_1)$  is still in  $P_1 \cap \ker \partial_0$ . Continue in a similar fashion to get  $\alpha_2, \alpha_3, \dots$ , until the resulting expression is 0. Let  $\alpha = \sum_i \alpha_i$ . Then  $x = \partial_1(\alpha)$  and  $\alpha \in P_2$ .

For (i), let  $x \in \ker \partial_0$ . Let  $\gamma \in P_2$  be a positive sum of length-2 cycles  $t_e$  with coefficients just large enough that  $x + \partial_1(\gamma) \in P_1$ . Then  $x + \partial_1(\gamma) \in P_1 \cap \ker \partial_0$ . By (ii), there exists  $\alpha \in P_2$  such that  $\partial_1(\alpha) = x + \partial_1(\gamma)$ . Then  $\partial_1(\alpha - \gamma) = x$ .  $\square$

For example, the element  $uv + vw - uw \in \ker \partial_0$  is  $\partial_1(\langle uvw \rangle - t_{uw})$ .

**Theorem 4.2** *Suppose  $G$  is  $(d, 3d/2)$ -colorable,  $d \geq 2$ . Then either*

- (i) *all simple cycles of  $G$  are short, in which case  $C^d = C$  and the quotient group  $C/C^d$  is trivial; or*
- (ii)  *$G$  contains a long simple cycle, in which case the quotient group  $C/C^d$  is isomorphic to  $\mathbb{Z}$ .*

*Proof.* Case (i) is immediate from Lemma 4.1(i). If all simple cycles are short, then  $H_2 = H_2^d$ , in which case  $C = C^d$ .

For case (ii), assume that  $G$  contains an oriented simple cycle  $T$  of length at least  $2d + 2$ . By Lemma 3.2,  $G$  has no simple cycle whose length lies between  $d + 1$  and  $2d + 1$ , inclusive. In this case  $C^d$  is generated by (edges of) simple cycles of length at most  $d$ .

We first show that  $\partial_1(T)$  is not in  $C^d$ , which implies that  $C^d \neq C$ , thus the quotient  $C/C^d$  is nontrivial. By general considerations, we know that it is a  $\mathbb{Z}$ -module of some finite dimension. We will show later that the dimension is 1.

Call an element of  $H_2$  *reduced* if it is an expression of the form  $\beta + \gamma$ , where

- (i)  $\beta \in P_2$ ;
- (ii)  $\beta$  is orthogonal to  $H_2^2$ ; that is,  $t_e \cdot \beta = 0$  for all  $e$ ;
- (iii)  $\gamma \in H_2^2$ ;
- (iv) it is not the case that  $\partial_1(t_e) \leq \partial_1(\beta)$  for any edge  $e$ .

Intuitively, conditions (i)–(iv) say, respectively, that  $\beta$  is positive,  $\beta$  is a sum of cycles of length 3 or greater,  $\gamma$  is a weighted sum of cycles of length 2, and no two cycles of  $\beta$  contain a complementary pair of edges  $e, \bar{e}$ .

We now claim that for every element of  $\alpha \in H_2$ , there is a reduced expression  $\beta + \gamma \in H_2$  such that  $\partial_1(\alpha) = \partial_1(\beta + \gamma)$ . Moreover, if  $\alpha \in H_2^d$ , then we can ensure that  $\beta \in H_2^d$ ; thus the reduction process does not introduce any long cycles.

First, express  $\alpha$  as a sum  $\beta_1 + \gamma_1$ , where  $\beta_1$  is a weighted sum of cycles of length at least 3 and  $\gamma_1$  is a weighted sum of cycles of length 2. Then  $\beta$  and  $\gamma$  satisfy (ii) and (iii). To obtain (i), we observe that

$$\partial_1(\alpha + \bar{\alpha}) = \partial_1\left(\sum_e (e \cdot \alpha)t_e\right);$$

intuitively, all the edges in  $\alpha + \bar{\alpha}$  occur in complementary pairs. Thus

$$\partial_1(-\alpha) = \partial_1(\bar{\alpha} - \sum_e (e \cdot \alpha)t_e).$$

If  $\beta_1 = \beta_1^+ - \beta_1^-$  with  $\beta_1^+, \beta_1^- \in P_2$ , let

$$\beta_2 = \beta_1^+ + \bar{\beta}_1^- \quad \gamma_2 = \gamma_1 - \sum_e (e \cdot \beta_1^-)t_e.$$

Then  $\partial_1(\alpha) = \partial_1(\beta_2 + \gamma_2)$  and  $\beta_2 + \gamma_2$  satisfies (i)–(iii).

Note that none of the operations so far have introduced any long cycles. Thus if  $\alpha \in H_2^d$  then  $\beta_2 \in H_2^d$ .

Finally, to get (iv), suppose  $\partial_1(t_e) \leq \partial_1(\beta_2)$ . Since  $t_e \cdot \beta_2 = 0$ , we must have  $\theta, \theta' \in \Delta$  and  $e \in \vec{E}$  such that  $\theta + \theta' \leq \beta_2$  and  $\partial_1(t_e) \leq \partial_1(\theta + \theta')$ . We have  $\partial_1(\theta + \theta' - t_e) \in P_1 \cap C$ , so by Lemma 4.1(ii) there exists  $\eta \in P_2$  such that  $\partial_1(\eta) = \partial_1(\theta + \theta' - t_e)$ . Write  $\eta$  as  $\eta' + \eta''$ , where  $\eta'$  is orthogonal to  $H_2^2$  and  $\eta'' \in H_2^2$ , and let

$$\beta_3 = \beta_2 - \theta - \theta' + \eta' \quad \gamma_3 = \gamma_2 + \eta'' + t_e.$$

Then  $\partial_1(\alpha) = \partial_1(\beta_3 + \gamma_3)$  and  $\beta_3 + \gamma_3$  still satisfies (i)–(iii).

Moreover,  $\beta_3 \in H_2^d$  if  $\beta_2 \in H_2^d$ . To see this, observe that the inner product with  $\sum_e e$  gives the sum of the coefficients, which for a cycle gives the length of the cycle. If  $\eta = \sum_i \eta_i$ , where each  $\eta_i \in \Delta$ , then because  $\theta, \theta' \in \Delta^d$ , we have

$$\begin{aligned} \sum_i \left( \left( \sum_e e \right) \cdot \partial_1(\eta_i) \right) &= \left( \sum_e e \right) \cdot \partial_1(\eta) \\ &= \left( \sum_e e \right) \cdot \partial_1(\theta + \theta' - t_e) \\ &= \left( \sum_e e \right) \cdot \partial_1(\theta + \theta') - \left( \sum_e e \right) \cdot \partial_1(t_e) \\ &\leq 2d - 2, \end{aligned}$$

therefore each  $\eta_i$  in the sum  $\eta$  is of length at most  $2d - 2$ . By Lemma 3.2, it is of length at most  $d$ .

The previous step can be repeated only finitely many times before (iv) becomes satisfied, because each time the inner product of the current  $\beta_i$  with  $\sum_e e$  decreases by two. We have shown that every  $\alpha$  is equivalent modulo  $\partial_1$  to a reduced  $\beta + \gamma$ . Moreover, if  $\alpha \in H_2^d$ , then  $\beta \in H_2^d$ .

Now we argue that if  $\alpha \in \Delta$  and  $\partial_1(\alpha) \in C^d$ , then  $\alpha \in \Delta^d$ . This implies that  $\partial_1(T) \notin C^d$ , since  $T \notin \Delta^d$ .

Suppose  $\alpha \in \Delta$  and  $\partial_1(\alpha) \in C^d$ . If  $\alpha$  is of length 2, there is nothing to prove, so assume  $\alpha$  is of length at least 3. There is a reduced form expression  $\beta + \gamma$  such that  $\partial_1(\alpha) = \partial_1(\beta + \gamma)$ ,  $\beta \in H_2^d$ , and  $\gamma \in H_2^2$ . For any edge  $e \in \vec{E}$ ,

$$(e - \bar{e}) \cdot \partial_1(\alpha - \beta) = (e - \bar{e}) \cdot \partial_1(\gamma) = 0,$$

since  $\gamma$  is a weighted sum of cycles of length 2; therefore

$$e \cdot \partial_1(\alpha) - \bar{e} \cdot \partial_1(\alpha) = e \cdot \partial_1(\beta) - \bar{e} \cdot \partial_1(\beta).$$

Because  $\alpha$  is a simple cycle of length at least 3, each of  $e \cdot \partial_1(\alpha)$  and  $\bar{e} \cdot \partial_1(\alpha)$  is either 1 or 0, and not both are 1. Also, since  $\beta$  satisfies (iv) of reduced expressions, one of  $e \cdot \partial_1(\beta)$  or  $\bar{e} \cdot \partial_1(\beta)$  must be 0. It follows by an easy case analysis that we must have  $e \cdot \partial_1(\alpha) = e \cdot \partial_1(\beta)$  in all cases. As  $e$  was arbitrary,  $\partial_1(\alpha) = \partial_1(\beta)$ . As  $\alpha$  is a simple cycle and  $\beta$  is a positive sum of simple cycles, we must have  $\alpha = \beta$ .

Now we argue that there is only one long simple cycle modulo short simple cycles. Let  $T$  and  $T'$  be two oriented long simple cycles. Let  $a$  and  $b$  be points on  $T$  of maximum distance from each other. Let  $p$  and  $q$  be local articulation points on  $T$  such that  $p$  is of distance at most  $d/2$  from  $a$ , moving in the forward direction on  $T$ , and  $q$  is of distance at most  $d/2$  from  $b$ , also moving in the forward direction. Then  $p$  and  $q$  are at least distance  $d + 1 - \lfloor d/2 \rfloor = \lceil d/2 \rceil + 1$  apart on  $T$ . Since local articulation points are not subtended by chords, removal of  $p$  and  $q$  disconnects the graph. Moreover, both  $T$  and  $T'$  must traverse  $p$  and  $q$ , otherwise we could construct a forbidden subgraph like the one illustrated in Fig. 2. Each of  $T$  and  $T'$  traverse  $p$  and  $q$  exactly once, since they are simple, and we can assume without loss of generality that they are oriented so as to traverse them in opposite directions.

Consider  $\partial_1(T' + T) \in C$ . We will show that this expression vanishes modulo  $C^d$ . Every edge  $e$  of  $T'$  is contained in a maximal segment of  $T'$  all of whose intermediate nodes do not lie on  $T$ . This segment constitutes a chord of  $T$  consisting of edges of  $T'$ . The endpoints of the chord are distinct and lie on both  $T$  and  $T'$ . The length of the chord is at least 1 and at most  $d/2$ , otherwise we could construct a forbidden subgraph. The length of the segment of  $T$  subtended by the chord is at least 1 and at most  $d/2$  for the same reason. Thus these two undirected segments together form a short simple cycle, which we orient in the opposite direction from  $T'$ ; that is, the cycle contains the edge  $\bar{e}$  for every edge  $e$  on the chord. The segment of the short cycle coinciding with  $T$  is oriented in either the same direction as  $T$  or the opposite direction. Let  $R$  be the set of short simple cycles obtained in this way.

Now let  $\gamma$  be a maximal sum of cycles of length 2 such that  $\partial_1(\gamma) \leq \partial_1(T' + T + \sum R)$ . We argue that  $\partial_1(T' + T + \sum R - \gamma) = 0$ . Consider each edge  $e \in \vec{E}$  separately. If neither  $e$  nor  $\bar{e}$  is an edge of  $T'$  or  $T$ , then  $e \cdot \partial_1(T' + T + \sum R - \gamma) = 0$ . If  $e$  lies on  $T'$  but neither  $e$  nor  $\bar{e}$  lies on  $T$ , then there is exactly one

element of  $R$  containing  $\bar{e}$  and none containing  $e$ . Since both  $e$  and  $\bar{e}$  appear in  $\partial_1(T' + T + \sum R)$  with coefficient 1,  $t_e$  appears in  $\gamma$ , and

$$e \cdot \partial_1(T' + T + \sum R - \gamma) = e \cdot \partial_1(T' + \sum R - \gamma) = 0.$$

It remains to calculate the coefficient of  $e$  and  $\bar{e}$  in  $\partial_1(T' + T + \sum R - \gamma)$  for  $e = uv$  lying on  $T$ . Suppose  $p, u, v, q$  occur on  $T$  in that order. Let  $P$  be the segment of  $T$  between  $p$  and  $u$  and let  $Q$  be the segment of  $T$  between  $v$  and  $q$ . The segment of  $T'$  from  $q$  to  $p$  traverses an odd number of chords subtending  $e$ , including possibly one of  $e$  or  $\bar{e}$  itself, and the traversals from  $Q$  to  $P$  and from  $P$  to  $Q$  must alternate, with one more traversal from  $Q$  to  $P$ . Each of these chords is responsible for a cycle of  $R$  containing either  $\bar{e}$  or  $e$ , depending on whether the chord goes from  $Q$  to  $P$  or from  $P$  to  $Q$ , respectively. So the coefficients of  $e$  and  $\bar{e}$  in  $\partial_1(\sum R)$  are  $k$  and  $k + 1$ , respectively, for some  $k \geq 0$ . This includes the case in which  $e$  or  $\bar{e}$  is an edge of  $T'$ , except the value of  $k$  is one greater. Adding in the  $e$  from  $T$ , we have that the coefficients of  $e$  and  $\bar{e}$  in  $\partial_1(T' + T + \sum R)$  are both  $k + 1$ . Since  $\gamma$  was chosen maximally,  $t_e$  appears in  $\gamma$  with coefficient  $k + 1$ , therefore the coefficients of  $e$  and  $\bar{e}$  in  $\partial_1(T' + T + \sum R - \gamma)$  are both 0.

Since  $e$  was arbitrary, we have  $\partial_1(T' + T + \sum R - \gamma) = 0$ , so  $\partial_1(T + T') = \partial_1(\gamma - \sum R) \in C^d$ , therefore  $\partial_1(T')$  is equivalent to  $-\partial_1(T)$  modulo  $C^d$ . We have shown that every element of  $C$  is equivalent modulo  $C^d$  to  $kT$  for some  $k \in \mathbb{Z}$ .  $\square$

## Acknowledgements

This work was supported by NSF grant CCF-0635028. Any views and conclusions expressed herein are those of the authors and do not necessarily represent the official policies or endorsements of the National Science Foundation or the United States government.

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