Shortest Weighted Path With Nonnegative Weights

Dijkstra's Algorithm

See Section 14.6.2 of the text.
In the last class we found the shortest path from a specific source node to every other node in the graph, measuring the length of a path by the number of its edges.

The algorithm was: give each node a value and a predecessor node. Give the source node a value 0 and all other nodes value INFINITY. Initially make all nodes have null for their predecessors.

Maintain a queue of nodes. Initially the source node is added to the queue. Perform the following steps until the queue is empty:

a) Remove the head of the queue. Call this node X.

b) For each outgoing edge from X to another node Y, if the value of Y is INFINITY, make the new value of Y be the value of X + 1, make the predecessor of Y be X, and add Y to the queue.
Dijkstra's algorithm covers the extension of this to graphs with non-negative edge weights. For this algorithm we will assume that all weights are non-negative. The next algorithm will be less efficient, but it will handle the case of negative weights.

We measure the "length" of a path by the sum of the weights on its edges. We want to find the shortest (cheapest) path from a source node S to every other node in the graph.

For example:
What made the unweighted case work was that we first put into the queue the only node whose value was 0, then all of the nodes whose value was 1, then all whose value was 2 and so forth. We know our algorithm works because if there was a shorter path to it we would have put it into the queue sooner.

We can treat this algorithm similarly. We first get the only node whose path has sum 0 (i.e, the source node). We then find the next minimum-cost node, then the next and so forth. We will use a priority queue to store our "working set" of nodes. Each time we finish with a node we put its children into the priority queue.
Here is why we can't have negative edge costs. When we remove an item from our priority queue, we need to be sure that we have its cheapest path. If we allowed negative edge weights, there might be a roundabout path we haven't explored yet that has a large negative weight, giving a path to the node that is cheaper that what we have found so far. In that case our algorithm wouldn't work.
For example, suppose we have this graph and our source node is A:

Initially our priority queue has only A with value 0.

Queue: {((A 0))}
We remove A from the queue, output it, and add the nodes at the end of A's outgoing edges to the queue

Queue: {(B 4) (F 5)}
Output: [(A 0)]

We remove the head of the queue, B, then add its children with their weights added to B's
Queue: {(F 5) (D 14) (C 5)}
Output: [(A 0) (B 4)]

The next cheapest paths are to F and C. Let's do F first:

Queue: {(D 14) (C 5)}
Output: [(A 0) (B 4) (F 5)]

Next come C, which has edges to D and E
Queue: {(D 14) (D 8) (E 8)}
Output: [(A 0) (B 4) (F 5) (C 5)]

If we take (D 8) next we get
Queue: {(D 14) (E 8) (E 10) (G 12) }  
Output: [(A 0) (B 4) (F 5) (C 5) (D 8)]
Here is a restatement of where we are:

Queue: {(D 14) (E 8) (E 10) (G 12) }
Output: [(A 0) (B 4) (F 5) (C 5) (D 8)]

Next comes (E 8).

Queue: {(D 14) (E 10) (G 12) (G 9) }
Output: [(A 0) (B 4) (F 5) (C 5) (D 8) (E 8)]
One more step finishes the process:

Queue: {\( (D\ 14)\ (E\ 10)\ (G\ 12) \) }

Output: \([ (A\ 0)\ (B\ 4)\ (F\ 5)\ (C\ 5)\ (D\ 8)\ (E\ 8)\ (G\ 9) ]\)

The remaining 3 nodes in the priority queue have already been visited and assigned their distances, so they are ignored when they are removed from the queue.
We have multiple copies of a node in the priority queue with different values because our queue doesn't automatically restructure when one of its values changes. Once the node has been removed from the queue we skip over any remaining copies of it when we remove the head of the queue.

How long does this algorithm take? We might add a node to the priority queue for each edge of the graph, so the size of the queue might be $|E|$. Each removal of the head of the queue takes $\log(|E|)$. Altogether, the running time is $O(|E| \log(|E|))$. This is slightly, but only slightly worse than for unweighted graphs.