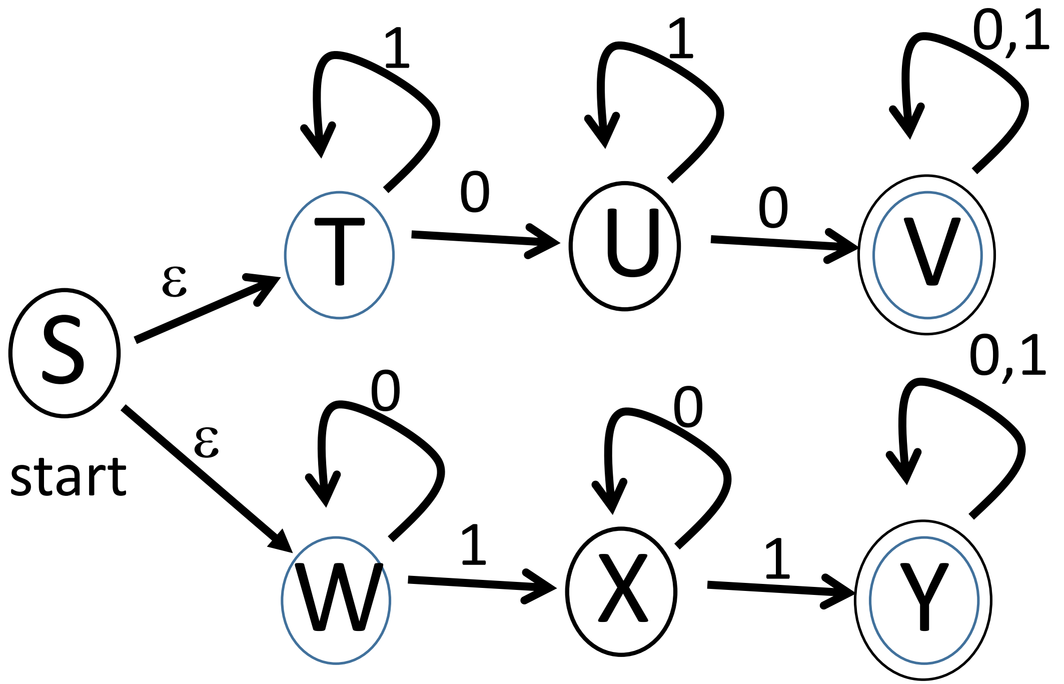
ε-NFAs

Here is another finite automaton -- an ε-NFA. Kozen calls these "NFAs with ε-transitions". These allow transitions labeled "ε" to be followed without consuming any input. These aren't interesting in themselves but are useful for showing that DFAs and regular expressions describe the same languages.

Example: Here is an e-NFA that accepts strings with 2 0's or 2 1's:



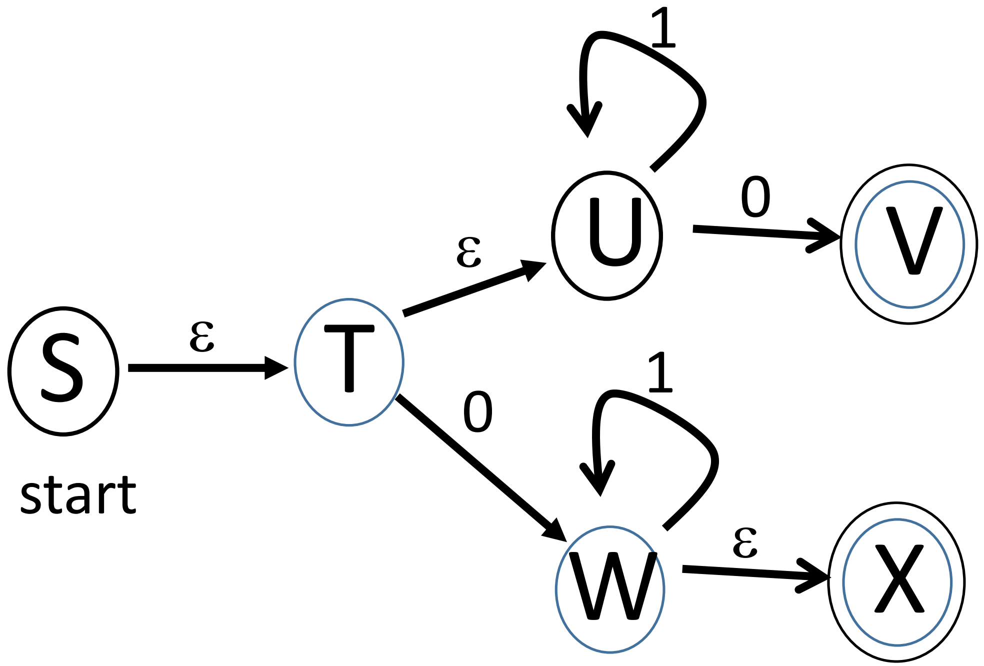
Formally, an ε-NFA is $(Q, \Sigma, \delta, s, F)$ where Q, Σ, s, and F are defined as with other NFAs and the inputs to δ are a state and either a letter in Σ or ε. This processes strings in the same way as the NFA $(\Sigma, Q, \delta', s, F)$, where $\delta'(q,a)=\delta(q,a) \cup \bigcup_{q' \in \delta(q,\varepsilon)} \delta'(q', a)$. (Note that this is a recursive definition of δ'.)

We are going to show that the language accepted by an ε-NFA is regular (so every ε-NFA has an equivalent DFA). To get there we need the idea of an *ε-closure* of a set of states. Let (Q, Σ, δ, s, F) be the ε-NFA we are talking about and let A be an set of states from Q. $\bar{A}$ will represent the closure of A. Here are two things we want to be true:

- A $\subset$ $\bar{A}$
- For each q in $\bar{A}$ if there is an ε-transition from q to q_1 then q_1 should be in $\bar{A}$.

This gives us an algorithm: to compute $\bar{A}$ start with A and add the destinations of ε-transitions until nothing else can be added.

Example:



This accepts 1^*0+01^* Here are some ε-closures:

$$\overline{\{T\}} = \{T, U\} \qquad \overline{\{U\}} = \{U\}$$

$$\overline{\{S\}} = \{S, T, U\} \qquad \overline{\{S, W\}} = \{S, W, T, U, X\}$$

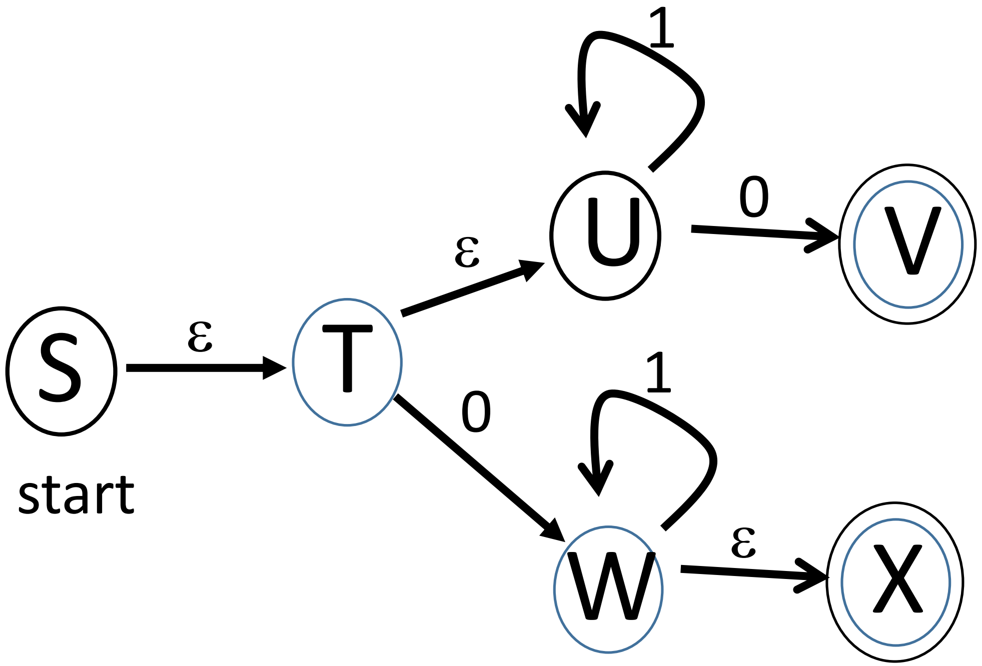
Theorem: Any language accepted by an ε-NFA is regular.

Construction: Let (Q, Σ, δ, s, F) be an ε-NFA. We will construct an equivalent DFA (Q', Σ, δ', s', F') :

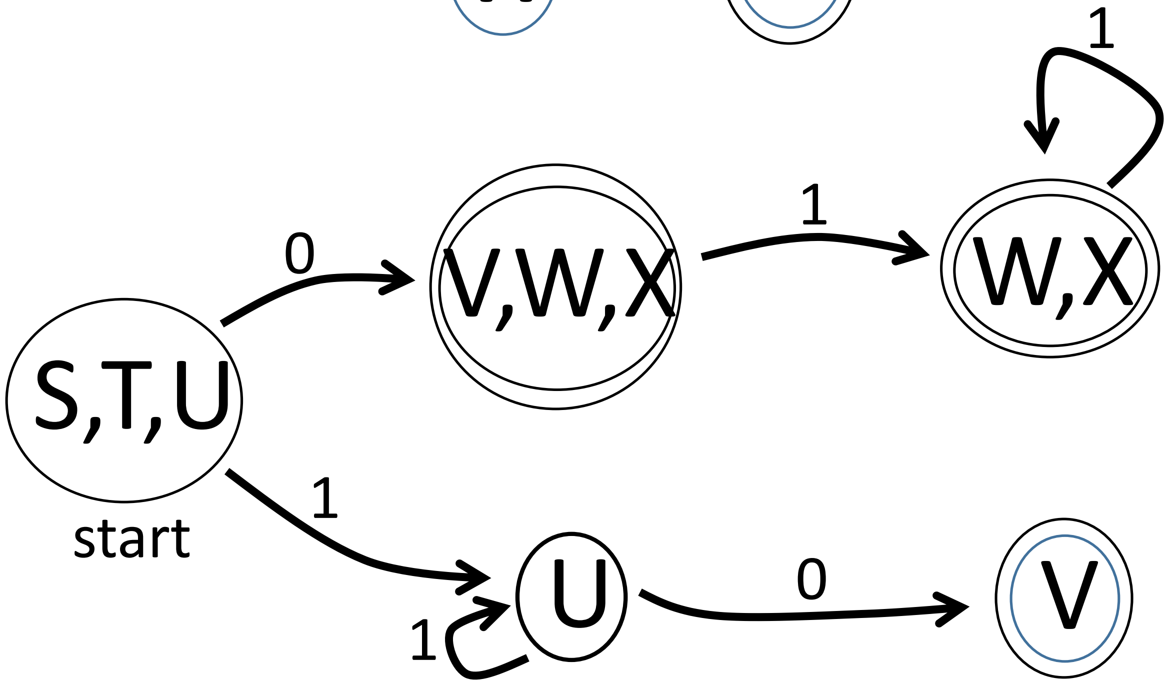
- Q' consists of subsets of Q
- s' = $\overline{\{s\}}$
- If P={q_0,q_1,...q_k} is a state in Q' and a is in Σ then P'=$\bigcup_{i=0}^{k} \overline{\delta(q_i, a)}$ is also a state in Q' and δ'(P,a)=P'
- If P={q_0,q_1,...q_k} is a state in Q' and if any of the q_i are in F, the P is in F'.

Example:

ε-NFA:

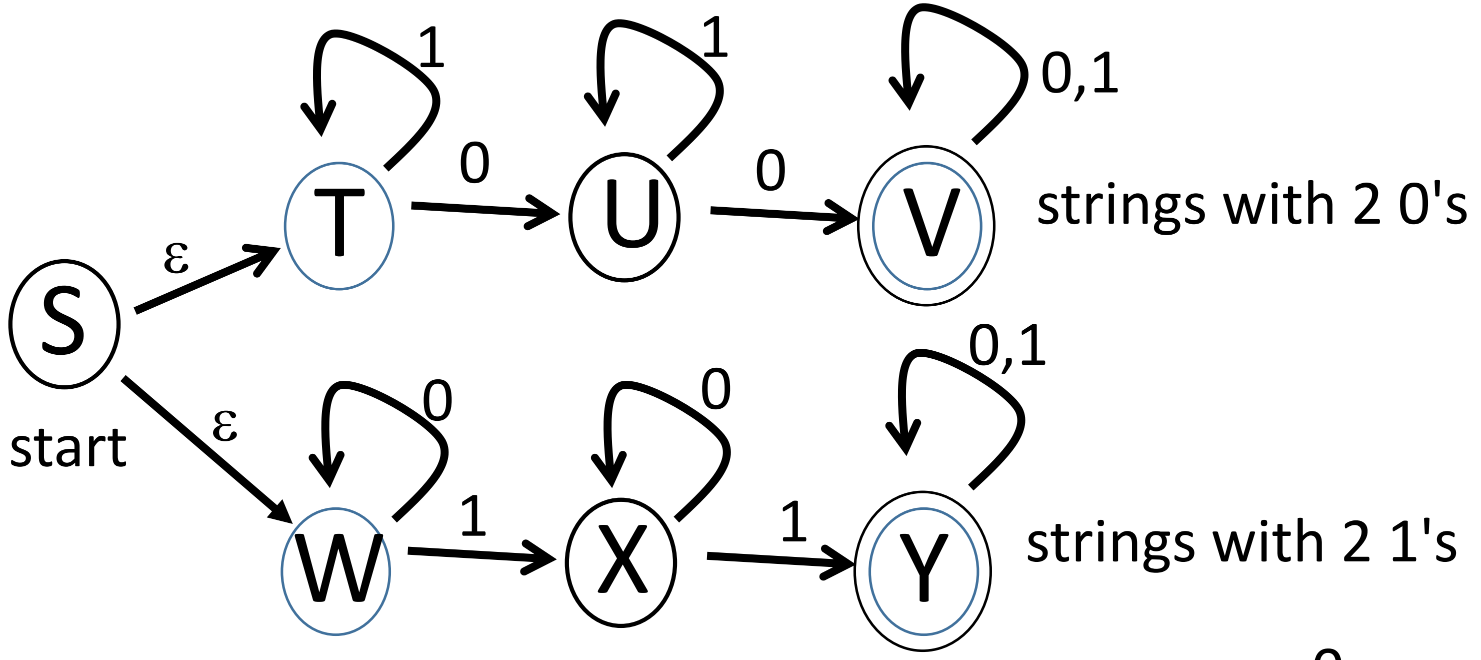


Equivalent DFA:

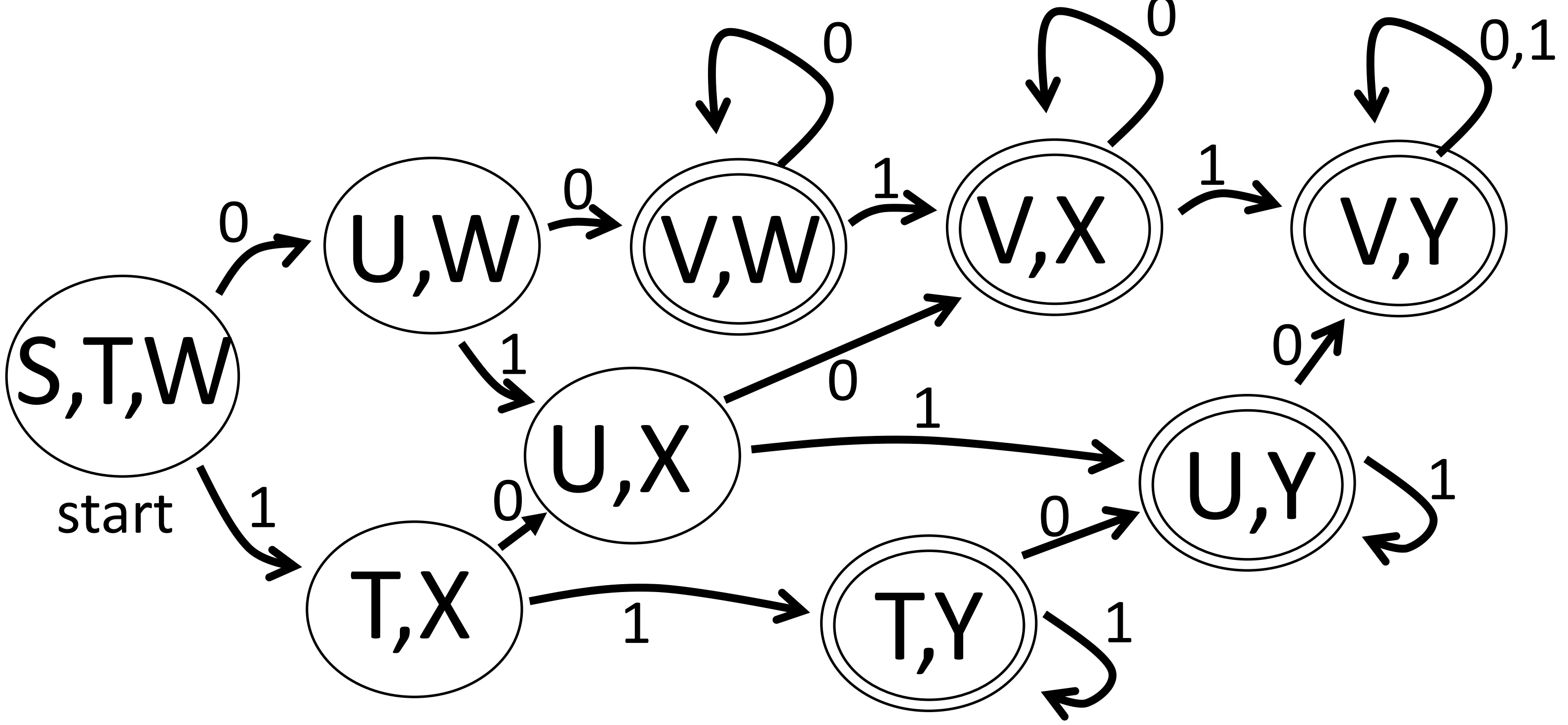


Example:

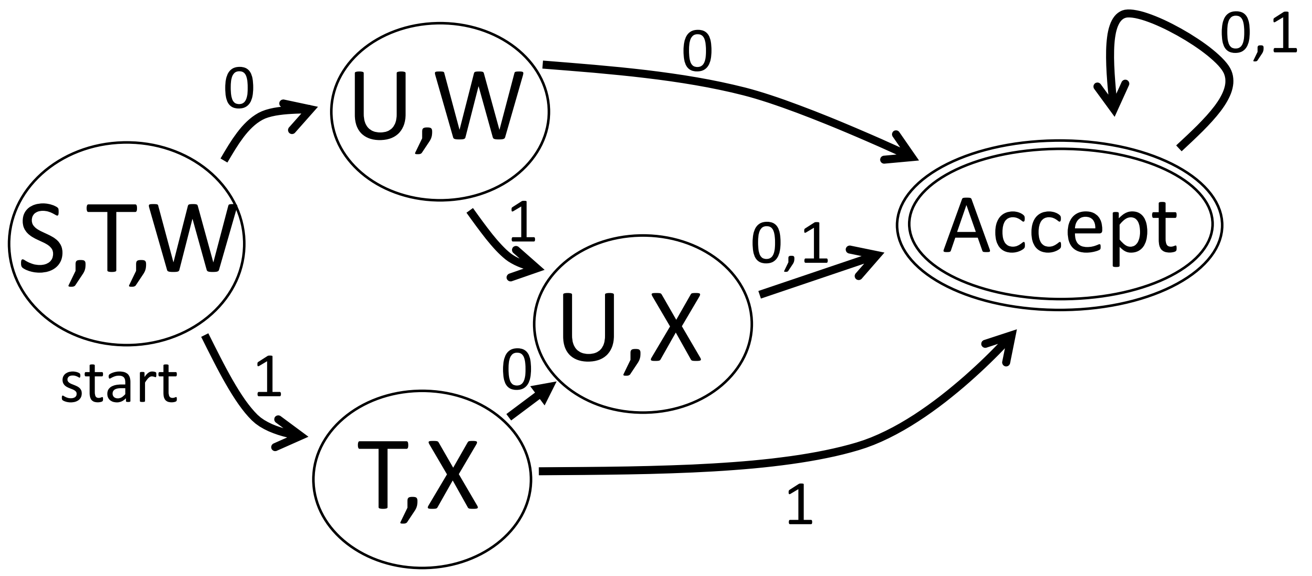
ε-NFA:



Equivalent DFA:



Note that this can be simplified to:



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We still need to prove that the DFA of this construction accepts exactly the same strings as the original ε-NFA. The proof is almost exactly the same as the proof that NFAs are equivalent to DFAs. If a string is accepted by the ε-NFA, processing the string takes the automaton through states q_0, q_1, ... q_k, where q_k is final. The q_i will be elements of states through which the DFA will pass while processing the string. The DFA will end in a state containing q_k, which is final, so it will accept the string.

Alternatively, if α is a prefix of the string at it takes the DFA to state $\{q_0,...q_j\}$ then on input α the ε-NFA could be in any of the q_i states. If the full string is accepted by the DFA it will also take the ε-NFA to an accept state.