# Pushdown Automata

A *pushdown automaton*, or PDA, extends the $\varepsilon$-NFA mode by adding a stack with its own alphabet $\Gamma$ (which may be different from $\Sigma$). Naturally, only the topmost symbol on the stack is visible.

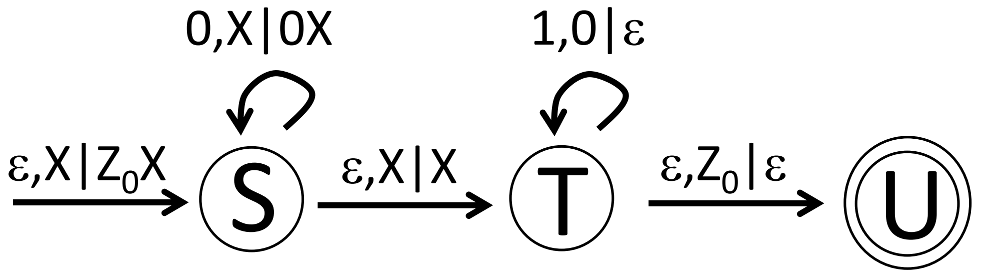
Transition notation for the stack:

$\xrightarrow{a,b|cb}$ means: on input a with b on top of the stack, push c (on top of b).

$\xrightarrow{a,b|c}$ means: on input a with b on top of the stack, pop the stack and push c.

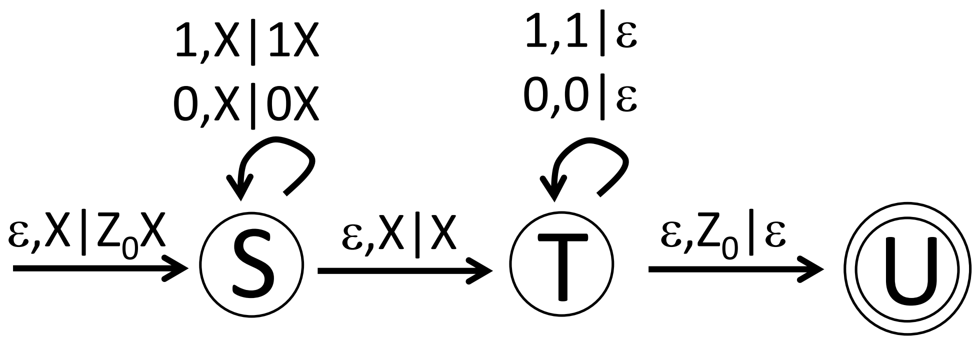
$\xrightarrow{a,b|\varepsilon}$ means: on input a with b on top of the stack, pop the stack.

Here is another example. This automaton accepts $\{0^n1^n | n>=0\}$



We will use the symbol $Z_0$ for the stack bottom (it marks the empty stack) and X as a placeholder for anything on the stack.

Example: the following PDA accepts strings in (0+1)* that are even-length palindromes



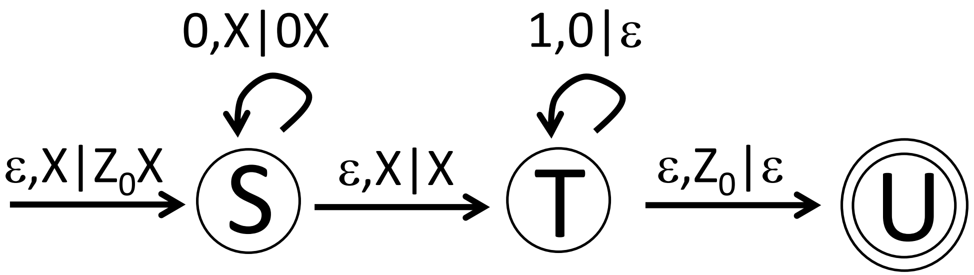

This automaton accepts strings that get to state U after consuming all of their input. Note that if it starts with an empty stack the stack will be empty at the end of the input.

We need a more formal and deterministic way to think about PDA computations.

A *configuration* is a triple (q, w, $\gamma$) where

- q is a state
- w is a string (the portion of the input not yet used)
- g is a string of stack symbols (the complete contents of the stack, with the top on the left)

Here is a configuration analysis of for the input 0011:




$(S,0011,Z_0) \rightarrow (T,0011,Z_0) \rightarrow (U,0011,Z_0)$

$\downarrow$

$(S,011,0Z_0) \rightarrow (T,011,0Z_0)$

$\downarrow$

$(S,11,00Z_0) \rightarrow (T,11,00Z_0)$

$\downarrow$

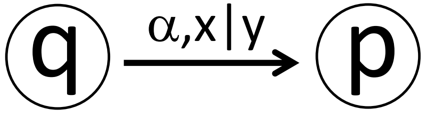
$(T,1,0Z_0)$

$\downarrow$

$(T,\varepsilon,Z_0) \rightarrow (U,\varepsilon,\varepsilon)$ accept

One step in such a configuration analysis is $(q, aw, x\beta) \longrightarrow (p, w, y\beta)$

This is valid if the PDA has a transition




We write $(q, w_1, \alpha) \xrightarrow{*} (p, w_2, \beta)$ if there is a sequence of steps that take the PDA from the first configuration to the second.

Formally a PDA is a 7-tuple $(\Sigma, Q, \delta, s, F, \Gamma, Z_0)$ where

- $\Sigma, Q, s, F$ have the same meanings as with DFAs
- $\delta$ is our configuration transforomation function
- $\Gamma$ is the alphabet of stack symbols
- $Z_0$ is the stack bottom

There are two commonly used definitions of what it means for the PDA to accept a string:

<u>Acceptance by final state</u>: If P is the PDA $(\Sigma, Q, \delta, s, F, \Gamma, Z_0)$ then $\mathcal{F}(P) = \{w \mid \text{there is } q \in F \text{ and } \alpha \in \Gamma^* \text{ so that } (s, w, Z_0) \xrightarrow{*} (q, \varepsilon, \alpha)\}$

<u>Acceptance by empty stack</u>: If P is the PDA $(\Sigma, Q, \delta, s, F, \Gamma, Z_0)$ then $\mathcal{E}(P) = \{w \mid \text{there is some state } q \text{ so that } (s, w, Z_0) \xrightarrow{*} (q, \varepsilon, \varepsilon)\}$

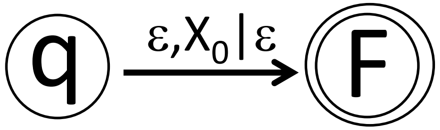
For a given automaton P, $\mathcal{E}$(P) and $\mathcal{F}$(P) are not necessarily the same. However, the languages that can be accepted by empty stack are the same as those that can be acceped by final state:

Theorem 1: Start with PDA P. Then there is a PDA P' where $\mathcal{F}$(P')=$\mathcal{E}$(P).

Theorem 2: Start with PDA P. Then there is a PDA P' where $\mathcal{E}$(P')=$\mathcal{F}$(P).

**Theorem 1**: Start with PDA P. Then there is a PDA P' where $\mathcal{F}$(P')=$\mathcal{E}$(P).

**Proof**: To make P', start with P. Create a new start state s' which pushes a new stack symbol $X_0$ onto the stack before $Z_0$. Make a new final state F. For each state q of P add a transition

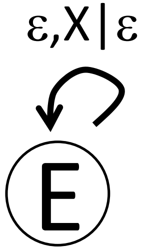



If P ever accepts a string by emptying its stack, P' can transition to its final state F.

On the other hand, if P' ever accepts a string then at the end of the the input there must be only $X_0$ on the stack, so P must have emptied its stack.

**Theorem 2**: Start with PDA P. Then there is a PDA P' where $\mathcal{E}(P')=\mathcal{F}(P)$.

**Proof**: This is easy. Start with P' the same as P. Give P' a new state E that empties the stack:




and add an ε-transition from every final state to E. String w can take P to a final state if and only if w empties the stack of P'.