Signed domination number of a graph and its complement.

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Abstract

Let $G = (V, E)$ be a simple graph on vertex set V and define a function $f : V \to \{-1, 1\}$. The function f is a signed dominating function if for every vertex $x \in V$, the closed neighborhood of x contains more vertices with function value 1 than with -1. The signed domination number of G, $\gamma_s(G)$, is the minimum weight of a signed dominating function on G. Let $\overline{G}$ denote the complement of G. In this paper we establish upper and lower bounds on $\gamma_s(G) + \gamma_s(\overline{G})$ and $|\gamma_s(G)||\gamma_s(\overline{G})|$.

Key words: signed domination, Nordhaus-Gaddum, graph complements.

1 Introduction

Let $G = (V, E)$ be a simple graph with $|V| = n$ and v a vertex in V. The *closed neighborhood* of v, denoted $N[v]$, is the set $\{u : uv \in E\} \cup \{v\}$. A function $f : V \to \{-1, 1\}$ is a *signed dominating function* if for every vertex $v \in V$, the closed neighborhood of v contains more vertices with function value 1 than with -1. We will use the symbol $f[v]$ to denote the sum $\sum_{x \in N[v]} f(x)$. Thus, f is a signed dominating function if $f[v] \geq 1$ for all $v \in V$. The weight of f, denoted $f(G)$, is the sum of the function value of all vertices in G, i.e., $f(G) = \sum_{x \in V} f(x)$. The *signed domination number* of G, $\gamma_s(G)$, is the minimum weight of a signed dominating function on G. This concept was defined in (1) and has been studied by several authors including (1; 2; 3; 4; 7). The (standard) domination number of a graph G, $\gamma(G)$, is similarly defined to be the minimum weight function $f : V \to \{0, 1\}$ such that $f[v] \geq 1$ for all $v \in V$. Given a function $f : V \to R$ we will say v is *dominated under* f, or simply *dominated*, if $f[v] \geq 1$

For any graph $G = (V, E)$ the complement $\overline{G} = (V, \overline{E})$ is defined to be the graph on the same set of vertices V with $uv \in \overline{E}$ if and only if $uv \notin E$, for all pairs $u, v \in V$. The notation $\overline{N[v]}$ will be used to denote the closed neighborhood of v in $\overline{G}$. Bounds on the sum or product of a parameter on a graph and its complement have been studied for several graph parameters starting with the Nordhaus-Gaddum results for the chromatic number (6). For standard domination it has been shown that for any graph G, $\gamma(G) + \gamma(\overline{G}) \leq n + 1$, with equality if and only if $G = K_n$ or $G = \overline{K_n}$. Furthermore, if both G and $\overline{G}$ are connected, then $\gamma(G) + \gamma(\overline{G}) \leq n$, with equality if and only if $G = P_4$ (see (5) (p. 237)). In the case of signed domination, however, the bounds are much weaker. Clearly, since $\gamma_s(G)$ is at most n, we have $\gamma_s(G) + \gamma_s(\overline{G}) \leq 2n$. We show that this trivial bound is in fact achieved in exactly six graphs, and more generally, we provide bounds on $\gamma_s(\overline{G})$ when $\gamma_s(G) = n$.

While the standard domination number is always positive, the signed domination number can be negative. In (4), we show a class of graphs whose signed domination number is arbitrarily close to $-n$. It is reasonable therefore, to look for a lower bound on $\gamma_s(G) + \gamma_s(\overline{G})$. In section 3 we give a lower bound. Finally, in section 4 a family of graphs which almost achieves the lower bound is presented as well as some other interesting examples.

2 Upper bounds on $\gamma_s(G) + \gamma_s(\overline{G})$

We present a lemma characterizing graphs G on n vertices for which $\gamma_s(G) = n$.

Lemma 1 *If G is a graph with $\gamma_s(G) = n$ then every vertex $v \in G$ is either isolated, an endvertex or adjacent to an endvertex.*

PROOF. Let G be a graph that contains a vertex v such that $deg(v) \geq 2$, and for each vertex y adjacent to v, $deg(y) \geq 2$. Consider the function $f : V(G) \to \{-1, 1\}$ for which $f(v) = -1$ and for any other vertex x, $f(x) = 1$. Clearly this is a signed dominating function. Therefore, the signed domination number of G is at most $n - 2$. Now suppose G is a graph with every vertex of degree 0 or 1 or adjacent to such a vertex. A signed dominating function must assign 1 to every vertex of degree 1, and it must assign 1 to every vertex that is adjacent to a vertex of degree 1. Hence, every vertex in G must be assigned a 1, and $\gamma_s(G) = n$.

Lemma 2 *If G is a graph containing a vertex of degree k then $\gamma_s(G) \geq 2 + k - n$.*

PROOF. Suppose $f : V(G) \to \{-1, 1\}$ and v a vertex of degree k in G. If f is to be a signed dominating function then $f[v] \geq 1$. The least possible weight for f will now be achieved if $f(x) = -1$ for all $x \notin N[v]$. In this case $f(G) = 1 - (n - (k + 1))$.

Theorem 3 *If G is a graph on n vertices such that $\gamma_s(G) = n$ then $0 \leq \gamma_s(\overline{G}) \leq 4$.*

PROOF. Let G be a graph with $\gamma_s(G) = n$. For the lower bound, observe that G must contain a vertex of degree 1 or 0. Therefore, $\overline{G}$ must contain a vertex of degree $n-2$ or $n-1$ respectively. Hence, by Lemma 2, $\gamma_s(\overline{G}) \geq 0$ or $\gamma_s(\overline{G}) \geq 1$.

For the upper bound, we exhibit a signed dominating function f on $\overline{G}$. The vertices will be labeled with respect to the edges in G where lemma 1 provides structure. Partition the vertices of G into two sets, T and H, such that T is an independent set with each vertex of degree 0 or 1 in G and every vertex in H is adjacent to at least one vertex in T. Notice that $|T| \geq |H|$. Label the vertices of H as h_i where $1 \leq i \leq |H|$. Partition T into the sets T_0, T_1, ..., $T_{|H|}$, where T_0 consists of all isolated vertices and for $i \neq 0$, T_i consists of all vertices in T adjacent to h_i. Label the vertices of T_i as $\{t_{i0}, \ldots t_{ip_i}\}$ where $p_i + 1$ is the number of vertices in t_i.

If $|H| = 0$ then G consists of isolated vertices and $\overline{G} = K_n$. It is clear that $\gamma_s(K_n) \in \{1, 2\}$. If $|H| = 1$, then G contains any number of isolated vertices and one star. Define f so that $f(h_1) = -1$, $f(T_0) \in \{0, 1\}$ and $f(T_1) \in \{2, 3\}$, where these depend on the parity of the respective sets. This signed dominating function for $\overline{G}$ shows that $\gamma_s(\overline{G}) \leq 3$.

Assume that $|H| \geq 2$. Initially, let $f(h_i) = -1$ for all $h_i \in H$; $f(t_{i0}) = +1$ for all $i > 0$. If $T_0 \neq \emptyset$ set $f(t_{00}) = +1$ and recursively define $f(t_{0j}) = -f(t_{0(j-1)})$ for $j = 1, \ldots p_0$, then set $f(t_{(i+1)1}) = -f(t_{i(p_i)})$ for $i \geq 0$ and $f(t_{ij}) = -f(t_{i(j-1)})$ for $i \geq 0, j \geq 2$. If $T_0 = \emptyset$, begin by setting $f(t_{11}) = +1$ and again proceed by recursively defining $f(t_{ij}) = -f(t_{i(j-1)})$ for $i \geq 1, j \geq 2$ and $f(t_{(i+1)1}) = -f(t_{i(p_i)})$ for $i \geq 2$. That is, the zeroth vertex in each set T_i, $i \geq 1$, gets $+1$ while all remaining vertices in T alternately get $+1, -1$ beginning with $+1$. At this time, $f(G)$ has a total weight of either 0 or 1, depending on the parity of $|H| - |T|$, but f is not necessarily a signed dominating function for $\overline{G}$. We will modify f based on the structure of G. This modification will only involve assigning $+1$ to two or fewer vertices which originally were assigned -1. Currently, $f(H) = -|H|$ and either $f(T) = |H| + 0$ or $f(T) = |H| + 1$. Consider any vertex t_{ij} in T. If $deg(t_{ij}) = 1$ in G, then in $\overline{G}$, t_{ij} is adjacent to every vertex but h_i. Therefore $f[t_{ij}] \geq 1$. If $deg(t_{0j}) = 0$ in G, then in $\overline{G}$, t_{0j} is adjacent to every other vertex, so $f[t_{0j}] \geq 0$. Therefore, as long as at least one -1 is changed to $+1$, all t_{ij} will be dominated under f in $\overline{G}$. Now we must ensure that the vertices in H are dominated as well. The modification of f will depend on value of $f(T)$ as well as the individual values of the $f(T_i)$. Note that $0 \leq f(T_i) \leq 2$, for $i \geq 1$ and $0 \leq f(T_0) \leq +1$. We define the *weight* of a set of vertices to be the sume of their values under the function f. Note that if $v \in S$ and we switch $f(v)$ from -1 to $+1$ then the weight of S increases by 2.

Case 1. $f(T) = |H|$, and there is no T_i set with weight equal to 2. In this case $f(T_i) = 1$ for all i. Pick any two vertices in H and switch their weights from -1 to 1. Every vertex in T is now dominated under f in $\overline{G}$. For any $h_i \in H$, $f(\overline{N[h_i]} \cap H) \geq 2 - |H|$ and $f(\overline{N[h_i]} \cap T) = f(T) - f(T_i) = |H| - 1$. Therefore, $f[h_i] \geq 1$ in $\overline{G}$ and the weight of f is 4.

Case 2. $f(T) = |H|$, and there is at least one T_k set with weight equal to 2. Then there must exist another set T_j with weight equal to 0. Switch the weight of a single vertex in T_j from -1 to 1, and switch the weight of h_j from -1 to 1 as well. Now, $f(\overline{N[h_j]} \cap H) \geq 2 - |H|$ and $f(\overline{N[h_j]} \cap T) = f(T) - f(T_j) \geq |H| + 2 - 2$, so that $f[h_j] \geq 2$ in $\overline{G}$. For any other vertex $h_i \in H$, $f(\overline{N[h_i]} \cap H) \geq 1 - |H|$ and $f(\overline{N[h_i]} \cap T) = f(T) - f(T_i) \geq |H|$. Therefore, $f[h_i] \geq 1$ in $\overline{G}$ and the weight of f is 4.

Case 3. $f(T) = |H| + 1$, and there is a vertex in $h_k \in H$ such that $\overline{N[h_k]} \cap H = H$. Switch the weight of h_k from -1 to 1. Then $f[h_k] = 3 - f(T_k) \geq 1$ in $\overline{G}$. For any other vertex $h_i \in H$, $f(\overline{N[h_i]} \cap H) \geq 2 - |H|$ and $f(\overline{N[h_i]} \cap T) = f(T) - f(T_i) \geq |H| - 1$. Therefore, $f[h_i] \geq 1$ in $\overline{G}$ and the weight of f is 3.

Case 4. $f(T) = |H| + 1$, and there is only one set T_k with weight equal to 2. Switch the weight of h_k from -1 to $+1$. Then $f(\overline{N[h_k]} \cap H) \geq 2 - |H|$, and $f(\overline{N[h_k]} \cap T) = f(T) - f(T_k) \geq |H| - 1$, hence $f[h_k] \geq 1$ in $\overline{G}$. For any other vertex $h_i \in H$, $f(\overline{N[h_i]} \cap H) \geq 1 - |H|$ and $f(\overline{N[h_i]} \cap T) \geq |H|$. Therefore, $f[h_i] \geq 1$ in $\overline{G}$ and the weight of f is 3.

Case 5. $f(T) = |H| + 1$, and there is more than one T_i set with weight equal to 2. There must be a set T_k with weight equal to 0. Switch the weight of a single vertex in T_k from -1 to 1. The weight of T_k is now 2 and $f(T) = |H| + 3$. For any vertex $h_i \in H$, $f(\overline{N[h_i]} \cap H) \geq -|H|$ and $f(\overline{N[h_i]} \cap T) = f(T) - f(T_k) \geq |H| + 1$. Therefore, $f[x] \geq 1$ in $\overline{G}$, and the weight of f is 3.

In section 4 we present infinite families of graphs for which $\gamma_s(G) = n$ and $\gamma_s(\overline{G}) = i$, for each of $i = 0, 1, 2, 3, 4$. We can now characterize graphs for which $\gamma_s(G) + \gamma_s(\overline{G}) = 2n$ as well as those for which $\gamma_s(G) + \gamma_s(\overline{G}) = 2n - 2$.

Theorem 4 *(i)*$\gamma_s(G) + \gamma_s(\overline{G}) = 2n$ *and* $\gamma_s(G)\gamma_s(\overline{G}) = n^2$*if and only if* $G \in \{P_1, P_2, \overline{P_2}, P_3, \overline{P_3}, P_4\}$, *where* P_i *is a path on* i *vertices.*

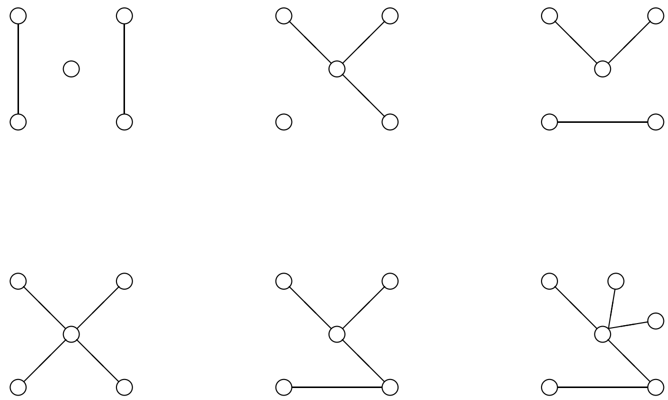

Fig. 1. All graphs on 5 or 6 vertices for which $\gamma_s(G) + \gamma_s(\overline{G}) = 2n - 2$.

(ii) $\gamma_s(G) + \gamma_s(\overline{G}) = 2n - 2$ *and* $\gamma_s(G)\gamma_s(\overline{G}) = n^2 - 2n$ *for exactly 12 graph/complement pairs.*

PROOF. Theorem 3 gives that $|V(G)| \leq 4$ for (i) and $|V(G)| \leq 6$ for (ii). The structure of lemma 1 allows us to quickly narrow the cases. For (ii) the explicit list of graphs includes K_3; all 5 acyclic graphs on 4 vertices; and the 5 graphs on 5 vertices and one graph on 6 vertices shown in Figure 1.

3 Lower bounds on $\gamma_s(G) + \gamma_s(\overline{G})$

Using known lower bounds on $\gamma_s(G)$ we can get a lower bound for $\gamma_s(G) + \gamma_s(\overline{G})$.

Theorem 5 *(4; 7) If G is a graph with maximum degree $\Delta(G) = \Delta$ and minimum degree $\delta(G) = \delta \geq 1$ then*

$$\gamma_s(G) \geq n\left(\frac{\delta - \Delta + 2}{\delta + \Delta + 2}\right).$$

A stronger form of this bound is given in (4). For convenience we will use A to denote the right hand side of this bound and B for the bound of lemma 2. Replacing Δ and δ by $(n-1-\delta)$ and $(n-1-\Delta)$ respectively, gives lower bounds for $\gamma_s(\overline{G})$ in terms of the parameters of G. These will be denoted by $\overline{A}$ and $\overline{B}$. Combining these bounds gives the next theorem.

Theorem 6 *For any graph G, $\gamma_s(G) + \gamma_s(\overline{G}) \geq -n - 2 + \sqrt{8n+1}$. Further, there is an infinite family of graphs for which equality holds.*

PROOF. There are four lower bounds for $\gamma_s(G) + \gamma_s(\overline{G})$, coming from the two bounds from theorem 5 and lemma 2 on the graph and its complement, namely $A + \overline{A}$, $A + \overline{B}$, $B + \overline{A}$, and $B + \overline{B}$. The second and third of these will have the same properties with the roles of G and $\overline{G}$ reversed.

For fixed n, and $0 \leq \delta \leq \Delta \leq n-1$, we want the minimum over all values of Δ and δ of the maximum of the 4 bounds, i.e.,

$$\min_{\delta,\Delta} [\sup(A + \overline{A}, A + \overline{B}, B + \overline{A}, B + \overline{B})].$$

This minimum must occur either (i) at a local minima for one of the bounds, (ii) on the boundary of the domain, or (iii) where 2 of the bounds intersect. We examine each of these cases in turn and then compare the results.

(i) Extrema. We consider the derivatives of the 4 bounds with respect to Δ and δ in order to find extrema.

$A+\overline{A} = n\left(\frac{\delta-\Delta+2}{\delta+\Delta+2} + \frac{\delta-\Delta+2}{2n-\Delta-\delta}\right)$. Taking the derivative with respect to Δ, setting equal to zero, and solving for Δ reveals no real solutions in our domain, since $n > \delta$.

$A + \overline{B} = n\left(\frac{\delta-\Delta+2}{\delta+\Delta+2}\right) + (1-\delta)$. The derivative of this with respect to Δ is always < 0 since n, δ are positive. Hence the function $(B + \overline{A})$ will also have no extrema.

$B + \overline{B} = 2 + \Delta - n + 1 - \delta$. This function is clearly always increasing with Δ and decreasing with δ.

Thus there are no local minima in the domain.

(ii) The Boundary. We reduce to one variable along the boundaries of the domain, $\delta = \Delta$, $\Delta = n-1$, and $\delta = 1$. In each case we must find the minimum of the maximum values.

$\Delta = \delta$: Here $A + \overline{A} = \frac{n}{1+\delta} + \frac{n}{n-\delta} \geq \frac{n}{1+\delta} + 1 \geq \frac{n}{1+\delta} + 1 - \delta = A + \overline{B} > 3 - n = 2 + \delta - n + 1 - \delta = B + \overline{B}$. That is, $A + \overline{A}$ gives the best (largest) lower bound in this case. It will be minimum when $\delta = \Delta = (n-1)/2$ at which point

$$\gamma_s(G) + \gamma_s(\overline{G}) \geq 4n/(1+n). \qquad (1)$$

$\Delta = n-2$: When $n - 4 \geq \delta$ then $B \geq A$. Further, as long $\overline{B} \geq \overline{A}$ then $B + \overline{B}$ is the largest lower bound. If $\overline{A} \geq \overline{B}$ then $B + \overline{A} =$ becomes the largest lower bound. Since $\overline{B}$ is always decreasing and $\overline{A}$ is always increasing the minimum case here occurs when $\overline{A} = \overline{B}$. This is at $\delta = \left(3 - \sqrt{1+24n}\right)/2 + n$. Here

$$\gamma_s(G) + \gamma_s(\overline{G}) \geq \left(-1 + \sqrt{1+24n}\right)/2 - n. \tag{2}$$

$\Delta = n-1$: The bound $\overline{A}$ does not directly apply as it requires $\delta(\overline{G}) \geq 1$. However, by assuming the complement will have at least one isolated vertex, and the minimum degree of the rest of the vertices is at least one, modified versions of bound A and B can be used. Analyzing these is similar to the case $\Delta = n-2$. The least value for this case is

$$\gamma_s(G) + \gamma_s(\overline{G}) \geq -n + (5 + \sqrt{24n-23})/2. \tag{3}$$

$\delta = 0$ and $\delta = 1$ are complementary to the above cases and so give the same minimum possible values.

(iii) The intersections. It suffices to consider the case where bound B equals bound A. This occurs when $\delta = (4n-4-4\Delta-\Delta^2)/(2+\Delta-2n)$. The function $B + \overline{A}$ will be strictly increasing as a function of Δ, while $B + \overline{B}$ reaches a maximum in the range $0 \leq \Delta \leq n-1$. Hence, the minimum of the maximum lower bounds in this case will occur either when $\Delta = 0, \Delta = n-1$, or when $\overline{A} = \overline{B}$. The first two cases were already considered. The last case occurs when $\Delta = 2(n^2-1)/(3n+1)$ at which point

$$\gamma_s(G) + \gamma_s(\overline{G}) \geq -2n(n-5)/(3n+1). \tag{4}$$

Comparison of cases. We compare the bounds for $\gamma_s(G)+\gamma_s(\overline{G})$ for each of the cases above, to determine the worst case lower bound. Comparing equations 1, 2, 3, and 4 reveal that for $n \geq 15$, the worst case occurs when $\Delta = n-2$ and $\overline{A} = \overline{B}$. Of course, this is equivalent to the case $\delta = 1$ and $A = B$.

The next phase is to show that no graph can exist with $\delta = 1, \Delta = (\sqrt{1+24n}-5)/2$, and $\gamma_s(G) = 2+\Delta-n$. Note that $n = (\Delta^2+5\Delta+6)/6$ will be an integer as along as $\Delta \equiv 0, 1 \pmod 3$ and this value of $\gamma_s(G)$ requires Δ is even. Assume $\Delta \equiv 0, 1 \pmod 3$ and even. Let x be a vertex of degree Δ in such a graph, G. By assumption there is a signed dominating function, $f : V(G) \to \{-1, 1\}$ with weight $f(G) = 2 + \Delta - n$. We will use this to determine the other edges of G. Let $Y = \{v \in V(G) | xv \notin E(G)\}$. Let $M = \{v \in V(G) | xv \in E(G)$ and $f(v) = -1\}$, $P = \{v \in V(G) | xv \in E(G)$, $f(v) = +1$ and $\deg(v) > 1\}$. Note that $f(y) = -1$ for each $y \in Y$. Hence all vertices of degree 1 must be adjacent to x and $f(x) = +1$. Hence $|M| = \Delta/2$, $|P| < \Delta/2$, and $|Y| = (\Delta^2 - \Delta)/6$. Now, for any $m \in M$, in order for $f[m] \geq 1$ it must be that m is adjacent to at least one vertex in P. Similarly, any $y \in Y$ must be adjacent to at least 2 vertices in P. Each $p \in P$ can be adjacent to at most $\lceil(\Delta-1)/2\rceil$ vertices of $M \cup Y$ (and an equal number from P). These adjacencies require that

$$\frac{\Delta}{2}\left\lceil\frac{(\Delta-1)}{2}\right\rceil \geq \frac{\Delta}{2}+\frac{(\Delta^2-\Delta)}{3}.$$

This inequality is false for all positive Δ. Hence such a graph can not exist.

Consider neighboring values of δ and Δ to see how close to this bound are actually attainable. The same argument as above shows that if a graph G with $\delta = 1$ has $\gamma_s(G) = 2 + \Delta - n$ then it must be that $|Y| \leq (\Delta^2 - 4\Delta)/8$ if Δ is even. In terms of n the smallest possible value for $\Delta = -2 + \sqrt{8n-4}$. Such a graph with $\delta = 1$ and $\Delta = -2 + \sqrt{8n-4}$ will have

$$\gamma_s(G) + \gamma_s(\overline{G}) \geq -n + \sqrt{8n-4}. \tag{5}$$

If Δ is odd the net result is slightly greater, $\gamma_s(G)+\gamma_s(\overline{G}) \geq 1-n+\sqrt{8n-7}$.

Additionally, we must consider the nearest (δ, Δ) pair along the line $A = B$. This occurs where $\delta = 2$, and $\Delta = \sqrt{1+8n} - 3$. In this case $\gamma_s(G) + \gamma_s(\overline{G}) \geq -n-2+\sqrt{8n+1}$. This is slightly less than the right hand side of equation 5 and hence is the lowest possible value for $\gamma_s(G) + \gamma_s(\overline{G})$. The Hajos graphs, H_k, have $n = k+k*(k-1)/2$, and $\gamma_s(H_k)+\gamma_s(\overline{H_k}) = (k-k*(k-1)/2)-1 = -n-2+\sqrt{8n+1}$, if n is odd. This example will be shown fully in section 4.

4 Examples of particular values.

While Theorem 4 gives an upper bound for $\gamma_s(G) + \gamma_s(\overline{G})$, it leaves open the question of the maximum value of $\gamma_s(G) + \gamma_s(\overline{G})$ for which there are an infinite number of graphs. In particular is there a value $r > 1$ such that for an infinite number of graphs, $\gamma_s(G) + \gamma_s(\overline{G}) > rn$? We leave this open, and provide instead some infinite families of graphs with other interesting values for $\gamma_s(G)$ and $\gamma_s(\overline{G})$.

Proposition 7 *There are infinite families of graphs for which*
(0) $\gamma_s(G) = n$ *while* $\gamma_s(\overline{G}) = 0$.
(i) $\gamma_s(G) = n$ *while* $\gamma_s(\overline{G}) = 1$.
(ii) $\gamma_s(G) = n$ *while* $\gamma_s(\overline{G}) = 2$.
(iii) $\gamma_s(G) = n$ *while* $\gamma_s(\overline{G}) = 3$.
(iv) $\gamma_s(G) = n$ *while* $\gamma_s(\overline{G}) = 4$

PROOF. (0) Take $V(G) = \{v_1, \ldots v_k, u_1, \ldots u_k\}$ with edges v_iv_j for all $1 \leq i \leq j \leq k$ and v_iu_i for all $1 \leq i \leq k$. This graph is sometimes called the corona of K_k. For (i) and (ii) take $\overline{G} = K_n$ since $\gamma_s(K_n) = 1, 2$ depending on the parity of n. For (ii) and (iii) take $K_{1,n-1}$. For (iv) take $V(G) = \{v_1, v_2, u_1, \ldots, u_{(4k+2)}\}$ and edges v_1v_2, v_1u_i for $1 \leq i \leq 2k+1$ and $v_2u_{(2k+1+i)}$ for $1 \leq i \leq 2k+1$, that is take two copies of $K_{1,2k+1}$ and add an edge joining the vertices of degree $2k+1$.

Proposition 8 *There is an infinite family of graphs for which*

$\gamma_s(G) = \gamma_s(\overline{G}) = 1.$

PROOF. For any positive integer k, there is a graph T_k on $4k+3$ vertices such that $\gamma_s(T_k) = \gamma_s(\overline{T_k}) = 1$. Let the vertices of T_k be $\{v_1, \ldots, v_{2k+1}, u_1, \ldots, u_{2k+1}, x\}$. The induced graph on $\{v_1, \ldots, v_{2k+1}\}$ is complete, as is the induced graph on $\{u_1, \ldots, u_{2k+1}\}$. For $k+1 \leq i \leq 2k+1$, (v_iu_i), (xu_i) and (xv_i) are edges of T_k. Hence for $1 \leq i \leq k$, $\deg(v_i) = \deg(u_i) = 2k$; for $k+1 \leq i \leq 2k+1$, $\deg(v_i) = \deg(u_i) = 2k+2$; and $\deg(x) = 2k+2$. Note that in the complement, $\overline{T_k}$, all vertices will also be of degree $2k$ or $2k+2$. By theorem 5, $\gamma_s(T_k) \geq 0$ and $\gamma_s(\overline{T_k}) \geq 0$. Since $|V|$ is odd for these graphs we get $\gamma_s(T_k) \geq 1$ and $\gamma_s(\overline{T_k}) \geq 1$.

A signed dominating function, f, for T_k is obtained by setting $f(u_i) = f(v_i) = -1$ if $1 \leq i \leq k$; $f(u_i) = f(v_i) = +1$ if $k+1 \leq i \leq 2k+1$; and $f(x) = -1$. A signed dominating function, g for $\overline{T_k}$ is obtained by setting $g(u_i) = g(v_i) = +1$ if $1 \leq i \leq k$; $g(u_i) = g(v_i) = -1$ if $k+1 \leq i \leq 2k$; $g(u_{2k+1}) = g(v_{2k+1}) = +1$; and $f(x) = -1$. As both $f(T_k) = g(\overline{T_k}) = 1$ these must be minimum weight signed dominating functions respectively. Hence $\gamma_s(T_k) = \gamma_s(\overline{T_k}) = 1$.

Proposition 9 *There are infinite families of graphs for which*

$\gamma_s(G) < 0$ *and* $\gamma_s(\overline{G}) = 0$ *and for which*

$\gamma_s(G) < 0$ *and* $\gamma_s(\overline{G}) = -1$

PROOF. The Hajos Graph, H_k, consists of the $k + k(k-1)/2)$ vertices, $V \cup U$, where $V = \{v_1, \ldots v_k\}$ and $U = \{u_{ij} : 1 \leq i < j \leq k\}$. The induced graph on V forms K_k, U is the empty graph and each u_{ij} is adjacent to v_i and v_j. The degree of each vertex in V is thus $2k-2$ and for each vertex in U is 2. Assigning $+1$ to each vertex in V and -1 to each vertex in U gives a signed dominating function with value $k - k(k-1)/2$. By Theorem 5 or Lemma 2 this is best possible. For the complement, Lemma 2 assures us that $\gamma_s(\overline{G}) \geq -1$. In fact for $k > 4$, $\gamma_s(\overline{H_k}) = 0, -1$ depending on the parity of $n = k + k * (k-1)/2$.

We exhibit a signed dominating function f on $\overline{H_k}$ with $f(\overline{H_k}) = 0, -1$. Define $f(v_i) = -1$ for all $v_i \in V$. Of the vertices in U, $\lceil (k+1)k/4 \rceil - k$ will also be assigned -1 and the other $\lfloor (k+1)k/4 \rfloor$ will be assigned $+1$. We determine which are which by constructing an auxiliary graph $A(X)$ on the vertex set $X = \{x_1, \ldots x_k\}$. This graph will have $m = \lceil (k+1)k/4 \rceil - k$ edges, with all vertices of degree $\lceil 2m/k \rceil$ or $\lfloor 2m/k \rfloor$. Any such graph $A(X)$ will do, and such a graph can be constructed by the classical theorem on the existence of graphs with given degree sequence.

Returning to $\overline{H_k}$, we now assign $f(u_{ij}) = -1$ if $(x_i x_j)$ is an edge in $A(X)$ and $f(u_{ij}) = +1$ otherwise. Each vertex in U is adjacent to all but two vertices in V and every other vertex in U. Hence

$$f[u_{ij}] = -(k-2) - (\lceil (k+1)k/4 \rceil - k) + \lfloor (k+1)k/4 \rfloor \geq 1.$$

Each v_i is adjacent to $(k-1)(k-2)/2$ vertices in U, namely all u_{kj} where $i \neq k$ and $i \neq j$. Of these, $m - \deg_{A(X)}(x_i)$ will have value -1. Some direct computation shows that

$$f[v_i] \geq -1 + (k-1)(k-2)/2 - 2\,(m - \lfloor 2m/k \rfloor) \geq k - 5.$$

The exact value of $f[v_i]$ depends on the value of k mod 4 as well as the degree of x_i in $A(X)$. Indeed, for $k = 5$, $f[v_i] = 1$. Hence for all $k \geq 4$, $f[v_i] \geq +1$ as required and f is a signed dominating function for $\overline{H_k}$.

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