Main Steps

The 4 main steps for proving a language A is not regular is as follows:

Step 1: Demon Picks $k \geq 1$. You are given some pumping length $k \geq 1$.

Step 2: You pick $xyz$. Select $x, y, z$ such that $xyz \in A$ and $|y| \geq k$.

Step 3: Demon Picks Decomposition $u, v, w$. The demon picks $u, v, w$ such that $y = u, v, w$ and $v \neq \epsilon$.

Step 4: You pick $i \geq 0$. Construct a string $xuv^i w z$ that is not in $A$, for some $i \geq 0$. Remember that you may want to set $i$ to 0 in order to accomplish this.

Comments

- The pumping lemma states a property of regular languages. You cannot use it to prove a language is regular, but you can use its contrapositive to prove a language is not regular. As a reminder, here is the pumping lemma in its positive form:

  If a language $A$ is regular, then there exists a $k \geq 1$ such that for all strings $x, y, z$ with $xyz \in A$ and $y \geq k$, there exist strings $u, v, w$ with $y = u, v, w$ and $v \neq \epsilon$, and for all $i \geq 0$, $xuv^i w z \in A$.

- Be sure your string $xyz$ is in $A$ and that $|y| \geq k$.

- Be sure to handle all possible decompositions of the string $y$ as $uvw$. The demon is picking this decomposition, and you cannot pick which decomposition he chooses.

- Don’t choose an $i$ that is fractional or negative! This is not allowed by the statement of the pumping lemma; $i$ must be an integer $\geq 0$.

- Your string $xyz$ should somehow depend on the pumping length $k$. If it doesn’t depend on $k$, then you cannot guarantee that it will be long enough for all possible values the demon provides.
Example

Consider the language $A = \{a^n b^n \mid n \geq 0\}$. We claim that $A$ is not regular.

Proof. We will show that $A$ is not regular using the contrapositive of the pumping lemma. That is, we will show that the pumping lemma properties do not hold, and therefore $A$ is not regular.

1. The demon chooses some pumping length $k \geq 1$.
2. We select $x = \epsilon$, $y = a^k$, and $z = b^k$. Then $xyz = a^k b^k \in A$, and $|y| \geq k$.
3. The demon now picks $u, v, w$ such that $y = uvw$ and $v \neq \epsilon$. Without loss of generality, suppose the demon picks $u, v, w$ of lengths $j, m, n$, respectively. Then $k = j + m + n$ and $m > 0$, and $y = a^j a^m a^n$.
4. But whatever the demon picks, we can win by taking $i = 2$:

   $$xuv^2wz = a^j a^{2m} a^n b^k = a^{j+2m+n}b^k = a^{k}a^m b^k,$$

   which is not in $A$ because there are different numbers of $a$’s and $b$’s.

\[\square\]