More Undecidability Examples

1. Let $L_{101}$ be the set of encodings of TMs that accept the string 101 and no other string. Is $L_{101}$ Recursively Enumerable?

Answer: No. Reduce complement of $L_u$ to it.

Given $(M, w)$ we create $M'$. $M'$ takes input $x$. If $x$ is 101, $M'$ accepts $x$. If $x$ is not 101 $M'$ ignores $x$ and simulates $M$ on $w$, accepting $x$ if $M$ accepts $w$.

If $M$ accepts $w$, $M'$ accepts all strings. If $M'$ does not accept $w$, $M'$ accepts only 101.

A recognizer for $L_{101}$ will recognize if $M$ does not accept $w$. Thus, a recognizer for $L_{101}$ creates a recognizer for the complement of $L_u$, and we know that can’t exist.

2. $L_{\text{inf}}$ is the set of encodings of TMs that accept infinitely many strings. Show $L_{\text{inf}}$ is not RE. Proof: We reduce the complement of the Universal language to $L_{\text{inf}}$. Let $(M, w)$ be a (TM, input) pair. Given $M$ and $w$, create $M'$: $M'$ takes input $x$ and simulates $M$ on $w$ for $|x|$ steps.

If $M$ halts and accepts $w$ in $|x|$ steps, $M'$ rejects $x$.

If $M$ halts and rejects $w$ within $|x|$ steps $M'$ accepts $x$.

If $M$ is still running after $|x|$ steps $M'$ accepts $x$.

If $M$ accepts $w$ in $n$ steps $M'$ rejects all $x$ with $|x| > n$, so $M'$ accepts only finitely many strings.

If $M$ does not accept $w$ and halts in $n$ steps, then $M'$ accepts all strings with length larger than $n$, so $M'$ accepts infinite many strings.

If $M$ does not accept $w$ and runs forever, $M'$ accepts all strings.

Altogether, $M'$ accepts finitely many strings if $M$ accepts $w$, and infinitely many if $M$ does not accept $w$. So if we could recognize if $M'$ accepts infinitely many strings, we could recognize if $M$ accepts $w$. 